Solving for the Nth Roots of Unity (Numerically)

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July 8, 2014

1 Conversion into a System of Polynomials

The Nth Roots of Unity come from the roots of the complex-valued polynomial

$$p(z) = z^n - 1 \tag{1}$$

Conversion of this complex-valued, analytic function into a system of two real-valued polynomials follows by setting z equal to $x + iy$ and seperating into real and imaginary parts. For example, in the case of $n = 2$, we get

$$z^2 - 1 = (x^2 - y^2 - 1) + (2ixy) \tag{2}$$

which corresponds to the system of polynomial equations

$$p(x, y) = \begin{cases} x^2 - y^2 - 1 \\ 2xy \end{cases}$$

Note that because $z^2 - 1$ has two roots at ± 1, we would expect to out system to have roots at $(\pm 1, 0)$. However, if we view our system as complex-valued instead of real-valued, we actually get four roots $(\pm 1, 0)$ and $(0, \pm i)$. One direct explanation is Bezout's Theorem. Another explanation is that by setting $z = (x + iy)$, can set x=0 and fix y accordingly, and vice versa.

In general, conversion of $z^n - 1$ leads to the following formula:

$$f(n) = \begin{cases} \sum\limits_{k=0}^{n} (-1)^{\frac{n-k}{2}} \binom{n}{k} x^k y^{n-k} & n - k \text{ even} \\ \sum\limits_{k=0}^{n} (-1)^{\frac{n-k+1}{2}} \binom{n}{k} x^k y^{n-k} & n - k \text{ odd} \end{cases}$$

This system has n real roots and $2n - 2$ complex roots. This is because we may assume that either x and y are both real, or that x is 0 and y is imaginary, or that y is 0 and x is imaginary, leading to n real roots and $2n$ complex roots. However, since 1 is always a root of unity, we've double counted it twice (since 1 is both a real root and an imaginary root) and so we subtract 2 from the total number of complex roots.