Competition Graphs and Permutation Patterns

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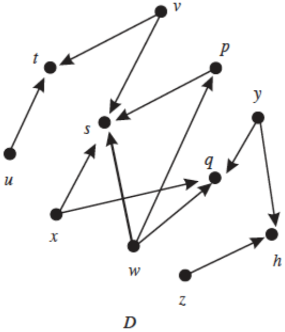
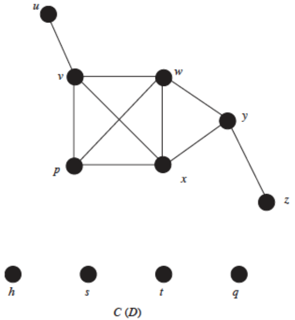
Competition Graphs

- Given digraph D, its **competition graph** $C(D)$ has:
  - The same vertex set $V(D)$
  - Edge between u and v if D contains arcs (u, w) and (v, w)



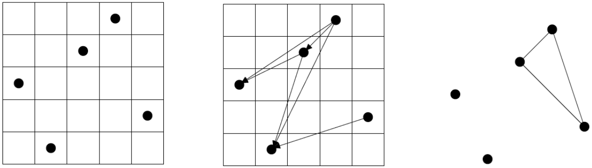


Doubly Partial Orders and Permutations

- The **doubly partial order** on $\mathbb{R}^2$:
  - $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$
  - $(x_i, y_i) \prec (x_j, y_j)$ if $x_i < x_j$ and $y_i < y_j$
- A **permutation** is a sequence $\pi = \pi_1\pi_2 \ldots \pi_n$, where each π_i is distinct and $\in \{1, 2, \ldots, n\}$
  - S_n is the set of all permutations of length n
  - A pattern $\tau \in S_k$, $k < n$, is the perm. on a subsequence of $\pi \in S_n$
  - Ex: $\pi = 5634127$ contains the pattern 123 but avoids 132
  - $S_n(\tau)$ is the set of $\pi \in S_n$ that avoid τ

The Competition Graph of a Permutation

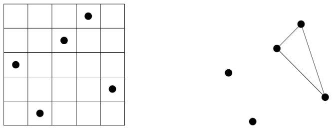
- Given $\pi \in S_n$, $D(\pi)$ is the digraph given by the doubly partial order.
- $C(D(\pi))$ is its competition graph.
  - For simplicity, use $C(\pi)$ instead of $C(D(\pi))$
- From left to right: $\pi = 31452$, $D(\pi)$, and $C(\pi)$



- **Edges in $C(\pi)$ correspond to 123, 132 patterns!**

Some More Notation

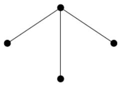
- Graph $G \in C(S_n)$, if there exists $\pi \in S_n$ such that $C(\pi)$, with the isolated vertices removed, is isomorphic to G
  - Analogously define $C(S_n(123))$ and $C(S_n(132))$
- If $\pi \in S_n$ is in the set $C_n^{-1}(G)$, $C(\pi)$, with the isolated vertices removed, is isomorphic to G
  - For instance, $31452 \in C_5^{-1}(K_3)$



  - Analogously define $C_n^{-1}(G, \tau)$, which restricts permutations to $S_n(\tau)$
- Let $c_n(G) = |C_n^{-1}(G)|$ and $c_n(G, \tau) = |C_n^{-1}(G, \tau)|$

Forbidden Induced Subgraphs

- Take any $G \in C(S_n)$:
  - If G contains P_3 as an induced subgraph, $G \notin C(S_{132})$
  - If G contains $K_{1,3}$ as an induced subgraph, $G \notin C(S_{123})$
- In other words (or notation):
  - $C^{-1}(P_3, 132) = \emptyset$ (No permutations in $S_n(132)$ can form P_3)
  - $C^{-1}(K_{1,3}, 123) = \emptyset$ (No permutations in $S_n(123)$ can form $K_{1,3}$)

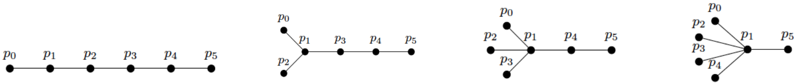


- Conjecture: Given that $G \in C(S_n)$:
  - $G \notin C(S_{132})$ **iff** P_3 is an induced subgraph
  - $G \notin C(S_{123})$ **iff** $K_{1,3}$ is an induced subgraph

Bijections

- From the previous slide:
  - $C^{-1}(P_3, 132) = \emptyset$
  - $C^{-1}(K_{1,3}, 123) = \emptyset$
- Can we do better?
  - $C^{-1}(P_m, 132) = \emptyset$
  - $C^{-1}(K_{1,3}, 123) = \emptyset$
- We can do even better!

$$C^{-1}(P_3, 123) \Longleftrightarrow C^{-1}(K_{1,3}, 132)$$
$$C^{-1}(P_m, 123) \Longleftrightarrow C^{-1}(K_{1,m}, 132)$$
$$C^{-1}(P_3) \Longleftrightarrow C^{-1}(K_{1,3})$$
$$C^{-1}(P_m) \Longleftrightarrow C^{-1}(K_{1,m})$$



Counting

- Formula for $c_n(K_{1,3}, 132)$, $c_n(K_{1,3})$ using induction

$$c_n(K_{1,3}, 132) = \sum_{k=1}^{n-6} [(k-1) \cdot 2^k + 1]$$

$$c_n(K_{1,3}) = 4 \cdot \sum_{k=1}^{n-6} [(k-1) \cdot 2^k + 1]$$

  - Problem: Induction doesn't generalize to $c_n(K_{1,m}, 132)$, $c_n(K_{1,m})$

- A different way to compute $c_n(K_{1,m}, 132)$, $c_n(K_{1,m})$
  - Insertion approach
  - Does not rely on induction
  - Problem: Insertion is conceptually easy but difficult in practice

Recurrence

- Solution: a recurrence relation:

$$c_n(K_{1,m}, 132) = c_{n-1}(K_{1,m}, 132) + c_{n-2}(K_{1,m-1}, 132)$$

m, n	3	4	5	6	7	8	9	10
1	1	4	12	32	80	192	448	1024
2	0	0	1	5	17	49	129	321
3	0	0	0	0	1	6	23	72
4	0	0	0	0	0	0	1	7

- Proof involves crafting a simple, but nontrivial bijection
- Allows us to compute $c_n(K_{1,m}, 132)$ for any m, n quickly
  - We only need the $n = 3, 4$ and $m = 1$ to complete the table

References

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A. Robertson, Permutations containing and avoiding 123 and 132 patterns, Discrete Math. Theoret. Comput. Sci. 3 (1999), 151154.

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