Network Flows

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REU 2014

June 6, 2014

Mentor
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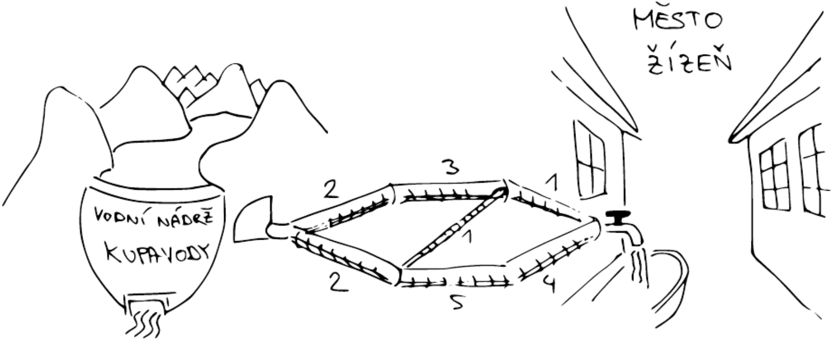
Source of all used pictures: Jakub Černý: The Basic Graph Algorithms

Motivation



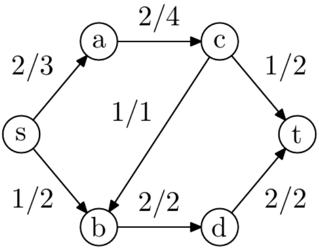
History and application

- Gustav Kirchhoff in the 19th century
- 1930 Tolstoi – solved problems around the transportation over the soviet railway, proved first lemmas
- 1955 Ford-Fulkerson algorithm
- Applications
  - Logistics
  - Airline scheduling
  - Mechanics
  - Biology

Formal definition

- Network is a tuple (V, E, s, t, c):
  - (V, E) directed graph
  - source $s \in V$, sink $t \in V$
  - capacity $c : E \to [0, \infty)$

- Flow in a network – function $f : E \to \mathbf{R}$ satisfying:
  - Capacity constraint: $\forall e \in E : 0 \leq f(e) \leq c(e)$
  - Kirchhoff's law: $\forall v \in V \setminus \{s, t\} : \sum_{e=(\cdot,v)} f(e) = \sum_{e=(v,\cdot)} f(e)$



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- *Integrality theorem*

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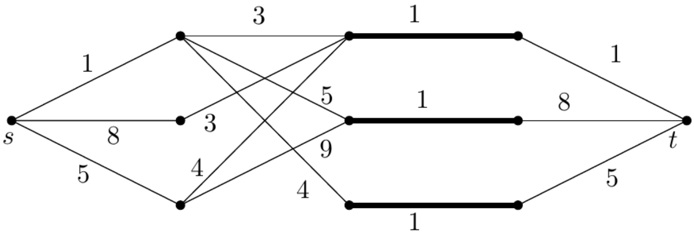
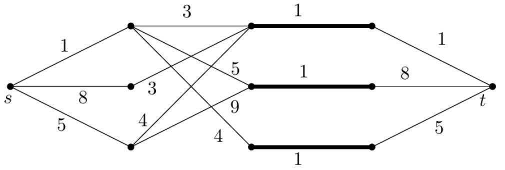
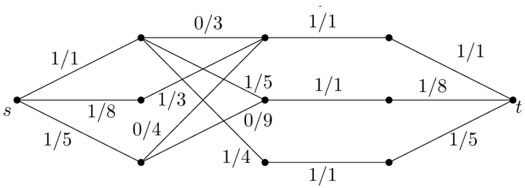


The duality of minimum cut and maximum flow

- *Ford-Fulkerson theorem:* Size of the maximum flow is equal to the size of the minimum cut.





Application: Bipartite matching

- Matchmaker's business – set of girls, set of boys
- Each person is attracted only to some other persons of the opposite gender
- *Task:* To make as many happy couples as possible.