$L(2, 1)$ - Labeling On Interval Graphs

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Abstract

Interval graphs are intersection graphs of a family of intervals of real numbers. $L(p, q)$-labeling of a graph G is a mapping $l\colon V_G \to X$ where $X \subset \mathbb{Z}$ such that $|l(u) - l(v)| \geq p$ whenever the vertices u and v are connected by an edge and $|l(u) - l(v)| \geq q$ whenever there exists some vertex w such that both u and v are neighbours of w. Finally, span of graph G is the smallest number k such that there exists $L(p, q)$-labeling of G using $X = \{0, \ldots, k\}$. We found a formula for the span of $L(2, 1)$-labeling for the class of interval graphs and its connection to the chromatic number of the graph and the maximum degree. We also mention, that $L(2, 1)$-labeling is NP-complete on interval graphs.

1 Introduction

A typical application of labeling is the channel assingment problem. There are some broadcasters and they use some frequencies, but if they are very close they must differ at least by a constant, in our case the constant is 2. Then there are broadcasters, which are close, but not very close, and they they must also use different frequencies, but the difference is less than in the case of very close broadcasters.

Before we start to describe the problem, let us give a formal definiton.

Definition 1. *$L(2, 1)$-labeling of a graph G(V,E) is a function $l : V \to \{0 \ldots k\}$ so that*

$$\{u, v\} \in E : |l(u) - l(v)| \geq 2\{u, v\}, \{v, w\} \in E : l(u) \neq l(w)$$

A span of a labeling of a graph is k and a span of graph $\lambda(G)$ is the smallest span over all correct labelings.

The problem of finding the minimal span was firstly anounced by Jerrold R. Griggs and Roger K. Yeh in [2]. They presented the first estimate for general graphs $\lambda(G) \leq \Delta^2 + 2\Delta$, but gave a conjecture $\lambda(G) \leq \Delta^2$, where Δ is the maximum degree of G. They prove the conjecture for 2-regular graphs. They also showed that this problem is NP-complete for general graphs.

Their estimate was later improved in [1] $\lambda(G) \leq \Delta^2 + \Delta$ They also gave exact values of the span for trees: $\lambda(G) = \Delta + 1$ or $\Delta + 2$.

We studied the problem for interval graphs, which are subclass of chordal graphs. *A chordal graph* is a graph, whose each cycle of length greater than 3 has a chord. In other words it does not cointain $C_{n\geq 4}$ as an inducted subgraph. Interval graphs are defined as intersection graphs. *An intersection graph* is a graph G(V,E), where V are some geometrical objects placed in R^d and there is an edge between two objects, if they intersect. And so *an interval graph* is an intersection graph of a family $\mathcal{F}$ of intervals on real line. *A unit interval graph* is an interval graph where all intervals have the same size.

Following estimate for the span of chordal graphs was given by Sakai in [3] and it is the only one, which was known for general interval graphs, too: $\lambda(G) \leq (\Delta + 3)^2/4$. This estimate proves the conjecture for chordal graphs.

By the same author the exact estimate for unit interval graphs was proved: $2\chi(G) - 2 \leq \lambda(G) \leq 2\chi$, where χ denote chromatic number of the graph G. Note that chordal graphs are perfect, so the chromatic number equals to the clique number.

We found tight bound for general interval graphs. The estimate is based on an algorithm, which use second power of the graph. As an easy observation based on the NP-completness for general graphs we show, that the problem remains NP-complete for interval graphs.

The paper is organized into five parts. In the first we describe the second power of an interval graph, then it follows part about the algorithm, which finds a correct $L(2,1)$-labeling. The next part derives the estimate from the algorithm and shows a graph, for which the bound is tight. In the last past we mention the reduction, which proves the $L(2,1)$-labeling is NP-complete on the interval graphs.

2 Power of Interval Graph

We starts with a formal definition of the graph power. Let have a graph $G(V,E)$. Then *kth-power of G* is $G^2(V, E^2)$ such that $E^2 = \{(u,v) : d(u,v) \leq k\}$, where $d(u,v)$ means the length of the shortest path connecting u and v. The following theorem and its proof describes, how to get the second power of an interval graph. It is obvious that this construction can be generalized for any k.

Theorem 2. *Let have $G(V,E)$ an interval graph and its representation. Then we can get G^2 in polynomial time and it is an interval graph, too.*

3 Algorithm for $L(2,1)$-labeling

As we have the representation of G^2, we can use it for an algorithm, which gives us a correct labeling of a given interval graph.

Theorem 3. *There exists an algorithm which gives us correct $L(2,1)$-labeling of a given interval graph. It runs in polynomial time.*

Note that this algorithm gives us a correct labeling, but it is not optimal. For example let have a this family of intervals: $\{\langle 0,1\rangle \langle 0,0\rangle, \langle 1,1\rangle, \langle 1,1\rangle\}$. The can be correctly labeled with numbers $0,3,2,4$, but the algorithm will find labels: $0,2,3,5$.

4 The Upper Bound

From the algorithm we are able to get the following estimate.

Theorem 4. *Let have an interval graph $G(V,E)$ of chromatic number χ and maximum degree Δ. Then*

$$\Lambda \leq 2\chi + \Delta - 2.$$

5 The Lower Bound

Observation 5. *The lower bound for the largest span is at least $\lambda = 2\chi + \Delta - 3$*

With this construction we can find an arbitrary large $\lambda(G)$ which is still very close to the estimate from the algorithm. But there is also another graph, which has the span even larger, but we cannot make it arbitrary large.

Observation 6. *There exist graphs with span $\lambda = 2\chi + \Delta - 2$*

Lemma 7. *There exist graphs with span $\lambda = 2\chi + \Delta - 2$ arbitrary large.*

All the presented examples has $\chi = 2$. We are not able to find any example with high span and $\chi > 2$.

6 NP-completeness

Jerrold R. Griggs and Roger K. Yeh prove in this paper that $L(2,1)$ - labeling is NP-complete for general graphs. It is easy observation based on their proof, that it remains NP-complete even on the interval graphs.

Theorem 8. *Problem of $L(2,1)$ - labeling is NP-complete on the interval graphs.*

7 Conclusion

We show tight bounds for the span of the interval graphs and show that the problem is NP-complete. We would like to see more examples of high span with arbitrary large χ. It may be also interesting to look at the graphs of general boxicity and try to find similar estimates.

References

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