

REU Problems of the Czech Group

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DIMACS

*Center for Discrete Mathematics and Theoretical Computer Science
Founded as a National Science Foundation Science and Technology Center*



DIMACS REU 2015, Piscataway

Complexity of Fair Deletion Problems

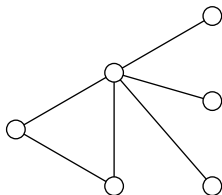
Graph Deletion Problems

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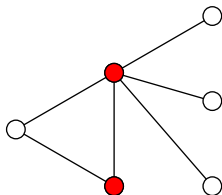
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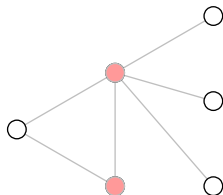
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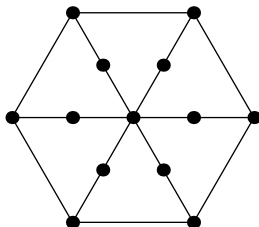
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The goal is to find the **smallest** such set.

Fair Versions of Deletion Problems

Sometimes the smallest set is not what we want.

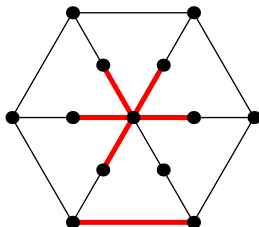
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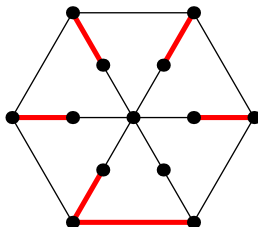


This solution removes a lot of links from single vertex.

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A much better solution.

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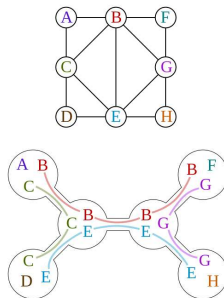
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For example, there is a parameter called **tree-width** that measures how much does the graph looks like a tree.

If we restrict ourselves to the graphs of **bounded** tree-width, we can get algorithms **polynomial** with respect to the size of a graph. (e.g. with running time $2^k n^3$, where k is the tree-width and n the size of a graph)

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- ▶ The result also works only for a specific fairness condition (number of deleted edges incident to a vertex), we want to explore other reasonable fairness conditions
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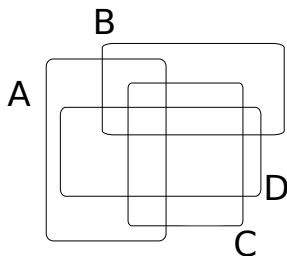
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- ▶ The current parameterization is by tree-width; we want to explore different parameterizations that will lead to FPT algorithms

Simplification of the Inclusion-Exclusion Formula

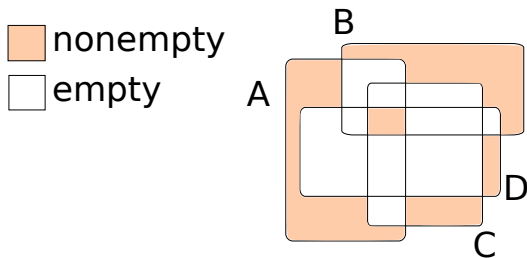
Inclusion-Exclusion Principle



$$\begin{aligned} |A \cup B \cup C \cup D| = & \\ & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D| \end{aligned}$$

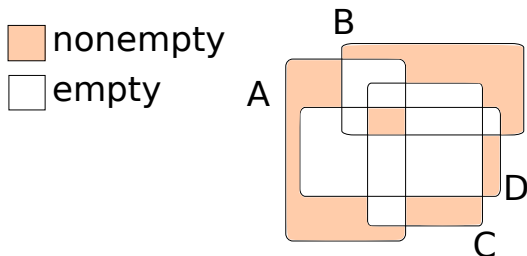
$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\emptyset \neq J \subseteq \{1, 2, \dots, n\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} A_j \right|$$

Simplified Formula



$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - 3|A \cap B \cap C \cap D|$$

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Short formulas can be found by methods from linear algebra.

Simplified Formula 2

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Problem: In practice, evaluating formulas with large coefficients amplify measurement errors.

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We want the formula

- ▶ to be short in the number of terms,
- ▶ to have small coefficients,
- ▶ to be polynomially derivable.

Research Problems

Considering formulas with coefficients from the set $\{-1, 1\}$.

Q: Given m nonempty regions what is the shortest formula?

Q: Is it NP-hard to decide whether there is a formula with at most k terms?

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Q: Given m nonempty regions what is the shortest formula?

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Known results (Tancer et al.):

Lower Bound: There are families of sets requiring $\Omega(m^{2-\varepsilon})$ terms.

Upper Bound: There is an algorithm which finds a formula of size $m^{O(\log^2(n))}$ in expected running time $m^{O(\log^2(n))}$.

Stronger results for specific families of sets (e.g. disks in a plane).

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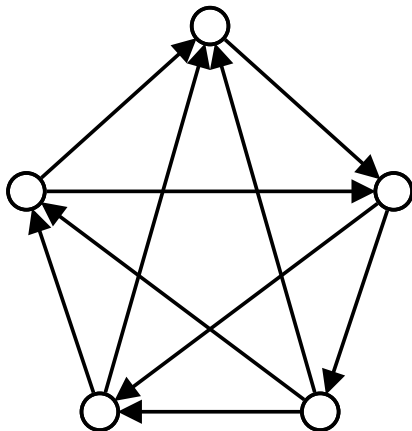
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- ▶ Complexity of the decision problem about existence of a short formula with $\{-1, 1\}$ coefficients.
- ▶ Studying structural properties of specific set systems described by Venn diagram regions.
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- ▶ Improving the upper bound.

Dominating Sets on Colored Tournaments

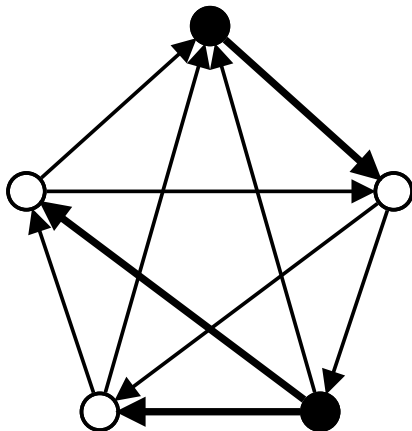
Dominating Set on Tournaments

- ▶ A **tournament** is a complete graph with orientated edges.
- ▶ A vertex v is **dominated by** u if there is an edge $u \rightarrow v$.
- ▶ A **dominating set** is a set of vertices called dominators such that every vertex of the graph is a dominator or is dominated by at least one.



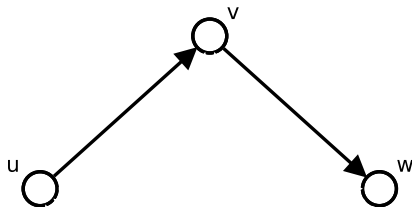
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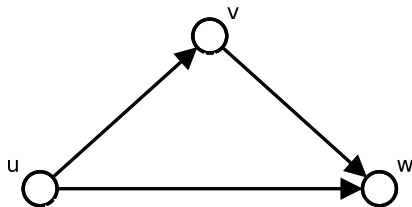
Edge-transitive Tournaments

- ▶ An **edge-transitive digraph**:
 $u \rightarrow v \& v \rightarrow w \implies u \rightarrow w$
- ▶ This means the graph must be **acyclic**.
- ▶ On transitive tournaments, one vertex dominates all.



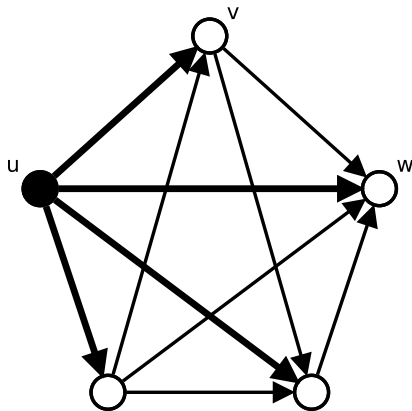
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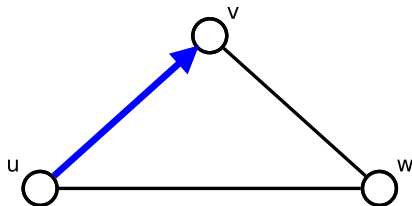
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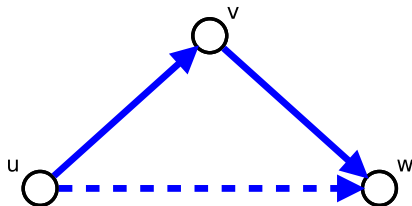
2-colored Transitive Tournaments

- ▶ Every edge is either red, or blue.
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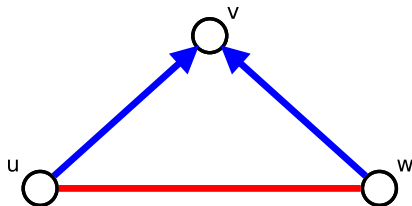
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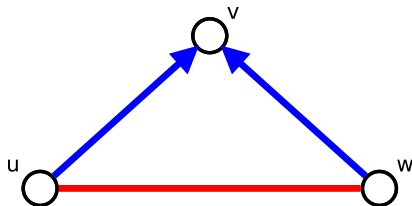
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- ▶ Then the whole graph is transitive and it has a single dominator.



k -colored Transitive Tournaments

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Plan

At first we will test small cases algorithmically, try various tournament reductions and then generalize the results for tournaments of an arbitrary size.

Subclasses of Chordal Graphs

Symmetries of Graphs

The study of symmetries of geometrical objects is an ancient topic in mathematics and its precise formulation led to group theory.

Symmetries of Graphs

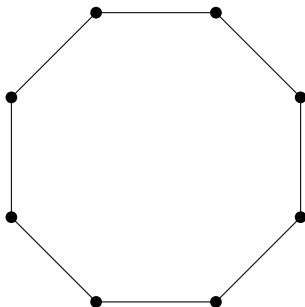
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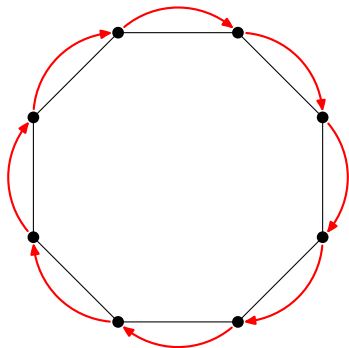
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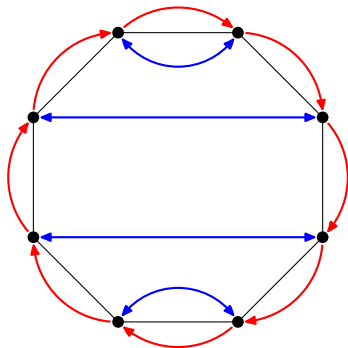
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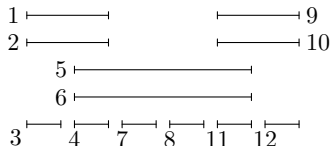
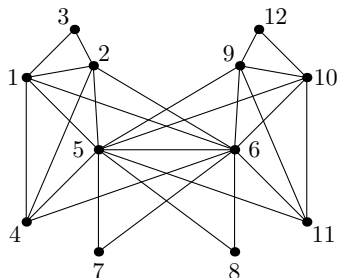
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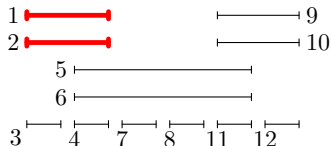
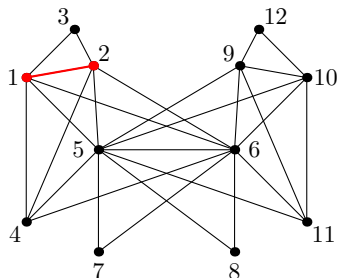
Interval Graphs (INT)

Interval graphs are intersection graphs of **intervals of the real line**.



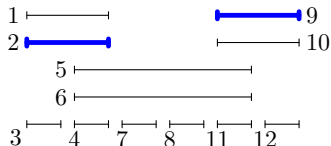
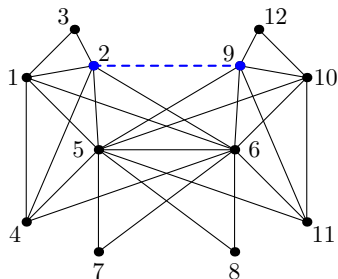
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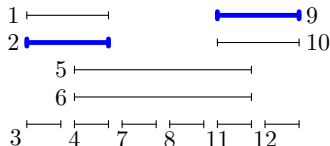
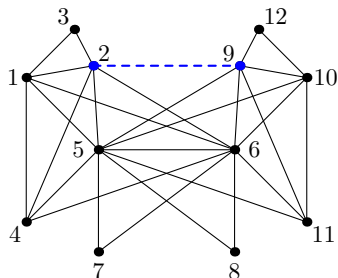
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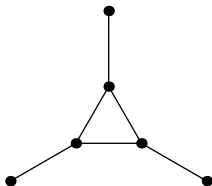
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Many computational problems are polynomially solvable for interval graphs (graph isomorphism, maximum clique, k -coloring, maximum independent set, ...).

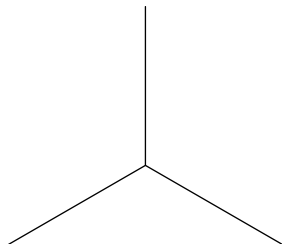
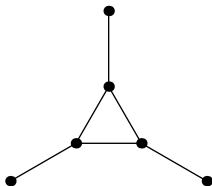
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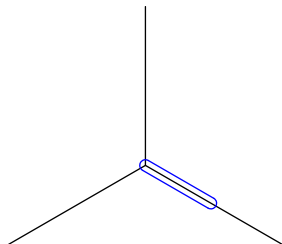
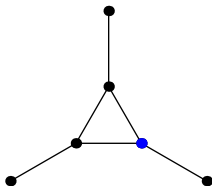
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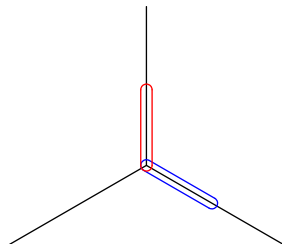
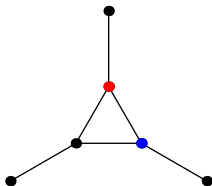
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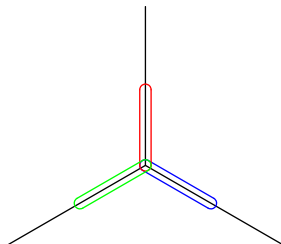
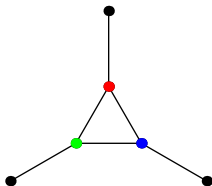
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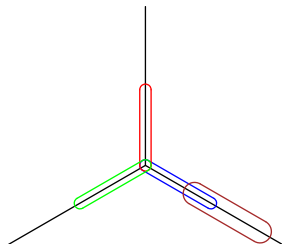
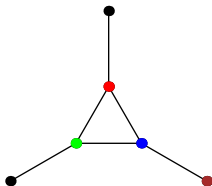
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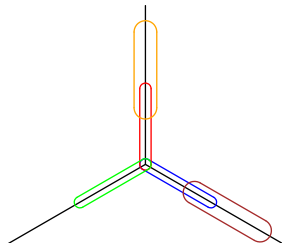
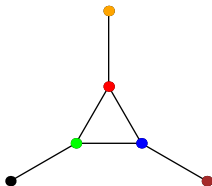
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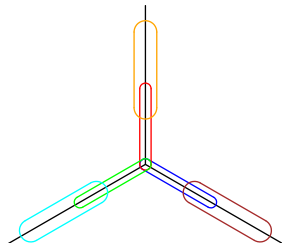
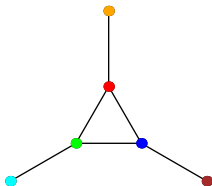
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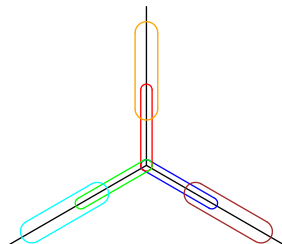
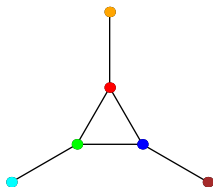
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It is easy to see that $\text{INT} \subsetneq \text{CHOR}$.

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Concerning the class T -GRAPH, not much is known in comparison to INT and CHOR.

Main Aim

Our **main aim** is to study the structural and computational properties of the classes T -GRAPH.

First, we will try to understand the classes T -GRAPH, for some special tree T (for example $K_{1,n}$).

It is possible that some techniques developed for interval graphs could be used.

Thank you!