# 1 Definitions

Let $G$ be a graph with one vertex marked as *destination* or just $d$.

Unformally, we want to decide whether it is possible to prescribe possible moves from a vertex to its neighborhood such that starting at arbitrary initial vertex we can find a walk to the destination even if some edges were removed.

A *switching table* for a vertex $v$ is a table with rows for every neighboring vertex. In the $u$'s row are *recommend* vertices. The first column it is recommend which neighboring vertex of $v$ we should move when we arrived to $v$ from $u$. In the $j$-th column, $j \geq 2$, it is recommend where to move when the edges to previous $j-1$ recommend vertex were removed. You can see an example of switching table for $K_4$ in Table 1.

*Remark* 1.1. The switching tables also can recommended from which edge to which edge to move – i.e. use edges instead vertices and problem still same.

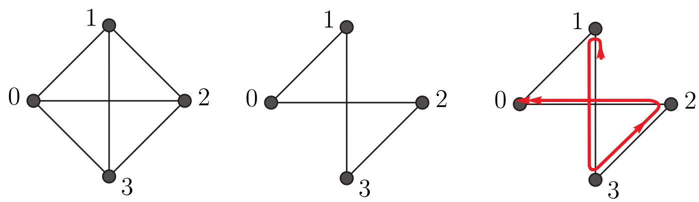


Figure 1: $K_4$; a ragged $K_4$'; a prescribed walk from vertex 1.

For edges $e_1, \dots, e_k$ of a graph $G$ a *ragged graph* $G'$ is $G \setminus \{e_1, e_2 \dots e_k\}$.

A *prescribed move* in $G'$ starting at an edge $(u, v)$ is an edge $(v, w)$. Where $w$ is the first vertex in $u$'s row in switching table for $v$ for which the edge $(v, w)$ is in $G$.

vertex 1

| neighbor | *switching table* | | |
|---|---|---|---|
| 0 | 0 | 2 | 3 |
| 2 | 0 | 2 | 3 |
| 3 | 2 | 3 | 0 |

vertex 2

| neighbor | *switching table* | | |
|---|---|---|---|
| 0 | 0 | 3 | 1 |
| 1 | 0 | 3 | 1 |
| 3 | 1 | 0 | 3 |

vertex 3

| neighbor | *switching table* | | |
|---|---|---|---|
| 0 | 0 | 2 | 1 |
| 1 | 0 | 2 | 1 |
| 2 | 0 | 2 | 1 |

Table 1: The switching tables for $K_4$. (0 is the destination)

A *prescribed walk* in $G'$ starting at an edge $(u, v)$ is a walk which use prescribed moves.

We say that a graph $G$ has a $k$-*resilient* routing if there are switching tables such that for every ragged graph obtained from $G$ by deleting $k$ edges and for every starting vertex there exists a prescribed walk which reach the destination.

A graph $G$ has a *routing* if it has a $(\mathrm{mincut}(G) - 1)$-routing, i.e. it has a $k$-resilient routing for the largest possible $k$.

## 1.1 Independent spanning trees

*Independent spanning trees* are directed spanning trees which are rooted in the destination and do not use same edge of $G$ in same direction. We often assign color to these trees. You can see an example of three independent spanning trees in Figure 2.

*Proof.* The graph $H$ is cubic with $n$ vertices, so it has $\frac{k \times n}{2}$ edges. $k$ of them, adjacent to destination, have just one one possible orientation and the remaining edges have two possible orientations. So there is

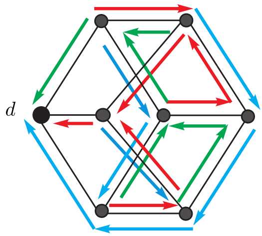


Figure 2: Independent spanning trees in a cube.

$k \times n - k$ possible places for edges of the spanning trees. Each spanning tree uses $n - 1$ edges. All $k$ of them thus use $k \times n - k$ edges, so every possible place is used. □

# 2 Reductions

We can prove that every 4-connected graph has four independent spanning trees using so called *reductions*.

**Reductions in general use the following idea** If a graph $G$ has some attribute, then we can use this attribute to construct a smaller graph $H$ or graphs $H_1$, $H_2$ which are still 4-connected. These reduced graphs have four independent spanning trees using induction. There are only a few possibilities how these trees use some specified subgraph of the reduced graph. For each of these possibilities we will construct four independent spanning trees of $G$.

## 2.1 Nontrivial $k$-cut reduction

A $k$-cut is *nontrivial* if it separates more than one vertex.

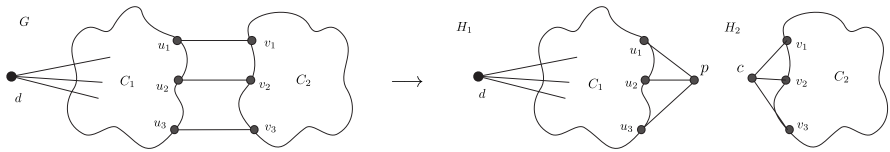


Figure 3: 3-cut reduction

**Reduction 2.1** ($k$-cut reduction)**.** *Let $G$ be $k$-connected graph with nontrivial $k$-cut composed by edges $(u_i, v_i)$ for $i \in (1 \dots k)$. A $k$-cut divides graph into components. Subgraph $C_1$ is the component which includes the destination and edges $u_i$. Subgraph $C_2$ are the other components.*

*The graph $H_1$ is created from $G$ by a contraction of subgraph $C_2$ into the vertex $p$. The graph $H_2$ is created from $G$ by a contraction of subgraph $C_1$ into the vertex $d$ and marking this vertex as the destination.*

**Lemma 2.2** ($k$-cut lemma)**.** *If a $k$-connected graph $G$ has a nontrivial $k$-cut and all smaller $k$-connected graphs have $k$ independent spanning trees, then $G$ also has $k$ independent spanning trees.*

*Proof.* The graphs $H_1, H_2$ are $k$-connected, because they were created from a $k$-connected graph by a contraction. The graphs $H_1, H_2$ have less edges then $G$ because $k$-cut was nontrivial. Thus $H_1$ and $H_2$ have $k$ independent spanning trees.

First, we color oriented edges of $G$ using independent spanning trees of $H_1, H_2$. Then we prove that every colored path ends in the destination and every vertex has exactly one outgoing edge of each color. Hence these colors determine independent spanning trees of $G$.

Edges in $C_1$ are colored by the same colors as they have in $H_1$. Edges $(u_i, v_i)$ are colored by the same colors as edges $(u_i, p)$ in $H_1$. We rename colors in $H_2$ such that the edges $(d, v_i)$ have the same color as the edges $(u_i, p)$. Note that these edges had only one direction colored in $H_2$. To edges in subgraph $C_2$ we assign the same color as they have in $H_2$.

We show that from every node $v$ we can use w.l.o.g. red edges to reach the destination. If $v \in C_2$, then there is a red path from $v$ to $d$ in recolored $H_2$. This path leads to $u_i$ and then using the red path in $H_1$ to the destination. If $v \in C_1$, then there was a red path in $H_1$. The problem is that this path can include the vertex $p$. In this case the path uses the edge $(u_i, p)$ which was replaced by edge $(u_i, v_i)$ and then continues via the red path from $v_i$ as described above. □

## 2.2 Useless edge reduction

An edge $e = (u, v)$ of the graph $G$ is *useless* if $\deg(u) > k$ and $\deg(v) > k$.

**Reduction 2.3** (Useless edge reduction)**.** *If $k$-connected graph $G$ has useless edge $e$ then $H = G \setminus \{e\}$.*

**Lemma 2.4** (Useless edge lemma)**.** *If a graph $G$ contains useless edge $e$ and has no nontrivial $k$-cut and all smaller $k$-connected graphs have $k$ independent spanning trees then $G$ also has $k$ independent spanning trees.*

*Proof.* First we will show that $H$ is $k$-connected. If $H$ has a $(k-1)$-cut $C$, then $C \cup \{e\}$ is a $k$-cut in $G$. A set $C \cup \{e\}$ is a nontrivial $k$-cut because the vertex in a trivial $k$-cut has degree $k$. $H$ has $k$ independent spanning trees and we can use exactly the same spanning trees for $G$. □

## 2.3 Multi-edge reduction.

**Reduction 2.5** (Multi-edge reduction)**.** *Let $G$ be a 4-connected graph with two edges $(u, v), \deg(u) = 4, \deg(v) = 5$. Then $H$ is created by contraction of $u$ and $v$.*

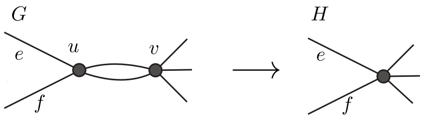


Figure 4: Multi-edge reduction.

**Lemma 2.6** (Multi-edge lemma)**.** *If a* 4*-connected graph* $G$ *contains multi-edge* $(u, v), \deg(u) = 4, \deg(v) = 5$ *and all smaller* 4*-connected graphs have* 4 *independent spanning trees then* $G$ *also has four independent spanning trees.*

*Proof.* The graph $H$ is 4-connected, because it was created from a 4-connected graph by a contraction.

Spanning trees of $G$ use same edges as in $H$ and a few edges are added.

At most two spanning trees leave $u = v$ in $H$ using edges $e, f$. These trees will use the multi-edge and remain connected. Then leaf nodes from $u$ are added to other trees so every spanning tree uses vertex $v$.

At most two spanning trees enter $u = v$ in $H$ using edges $e, f$. These trees will use the multi-edge and remain connected. Then leaf edges from $u$ are added to other trees so every spanning tree uses vertex $u$.

See 5

□

**Note** If $u$ or $v$ is a goal reduction works in the same way. Just the goal cannot be added as a leaf node.

## 2.4 Reduction of a vertex with four different neighbours.

**Reduction 2.7** (Vertex reduction)**.** *Let* $G$ *be a 4-connected graph without nontrivial 4-cut with vertex* $v$ *which has four neighbours*

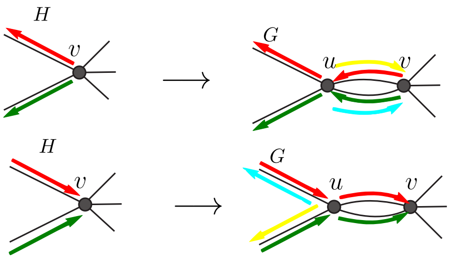


Figure 5: Independent spanning trees near a multi-edge.

$a, b, c, d$.

$$H = G \setminus \{v\} \tag{1}$$
$$H_1 = H \cup \{(a, b), (c, d)\} \tag{2}$$
$$H_2 = H \cup \{(a, c), (b, d)\} \tag{3}$$

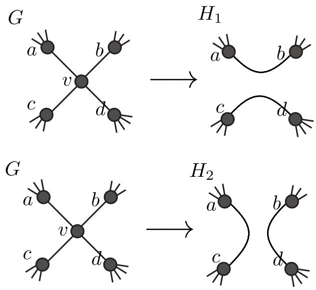


Figure 6: Reduction of vertex with four different neighbours.

**Observation 2.8.** *$H_1, H_2$ and $H$ are 3-connected.*

*Proof.* If there is 2-cut $\{f, g\}$ in $H_i$ and not in $G$ this 3-cut separates one or two vertices from $\{a, b, c, d\}$. If 2-cut separates $a$ then

$\{f, g, (v, a)\}$ is 3-cut in $G$. If 2-cut separates $a, b$ then $\{f, g, (v, a), (v, b)\}$ is nontrivial 4-cut in $G$. □

**Lemma 2.9.** *At least one of graphs $H_1, H_2$ is 4-connected.*

*Proof.* For contradiction, suppose that none of them is.

We define auxiliary graph $P$. 3-cut in $H_1$ divide vertices into groups $V_{1a}, V_{1b}$ and 3-cut in $H_2$ divide vertices into groups $V_{2a}, V_{2b}$. $P$ has four vertices which are created by contraction $v_1 = V_{1a} \cap V_{2a}, v_2 = V_{1a} \cap V_{2b}, v_3 = V_{1b} \cap V_{2a}, v_4 = V_{1b} \cap V_{2b}$. $P$ has at most six edges because all edges in $P$ are in mentioned cuts.

$P$ is not $K_4$ because $K_4$ is 4-connected. Without loss of generality there is no edge between $v_1$ and $v_2$. $P$ was created from 3-connected graph by contraction so it is also 3-connected and there are three edge independent paths between $v_1$ and $v_2$. Each of them is exactly two edges long. If three paths use $v_3$ then $v_4$ is not connected with the rest of the graph. If two paths use $v_3$ and one use $v_4$ then there is 2-cut around $v_4$. Contradiction.

□

**Lemma 2.10** (Vertex reduction lemma)**.** *If a 4-connected graph $G$ with no nontrivial 4-cut has vertex $v$ with degree 4 and all smaller graphs have four independent spanning trees, then $G$ also has four independent spanning trees.*

*Proof.* From lemma 2.9 we can suppose that $H_1$ is 4-connected and has four independent spanning trees.

We will describe spanning trees on edges incident with $v$. On rest of graph spanning trees use same edges both in $G$ and $H$. For simplification we suppose that both direction of edges $(a, b), (c, d)$ are used.

We subdivide edges $(a, b), (c, d)$. When they use four different colors we have correct spanning trees of $G$ but otherwise there is more then one outgoing edge with the same color from $v$. This problem can occur on two or four outgoing edges. In that case we choose the edge from which path continues directly to $d$ and add it to $G$. In place of the other edge we add leaf edges of spanning trees which do not include $v$ yet. In such way we get correct independent spanning trees.

□

## 2.5 Proof

**Theorem 2.11.** *Every 4-connected graph $G$ is 3-resilient.*

*Proof.* We prove the statement by induction on the number of edges.

First of all, if the graph has a nontrivial 4-cut, then we use the nontrivial 4-cut reduction. If $G$ contains an edge $e = (u, v)$ such that $\deg(u) > 4$ and $\deg(v) > 4$ we use the useless edge reduction. Between two vertices are at most two edges. If there is more then there is a 3-cut around these two vertices. If there is two edges between two vertices we use multi-edge reduction. If $G$ is $K_5$ it has four independent spanning trees as you can see 7. If $G$ has no useless edge and it is not $K_5$ it has vertex with degree four and we can use vertex reduction.

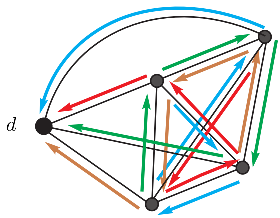


Figure 7: Four independent spanning trees for $K_5$.

□