

Report

Kenigsberg Lea

August 22, 2015

1 Motivation

When a rigid surface is vibrated at a resonant frequency, regions of zero displacement are formed. These regions are called nodal lines. Typically the surface is covered with sand or other fine particles that migrate to the nodal regions during the vibration, and result in a Chladni pattern. The motivation for this project arose from the following question: given a specific facial pattern, can we find a surface that will have this pattern as its nodal domain? In order to answer this question I studied Chladni patterns and looked at the properties and geometry of nodal regions. In this communication I will summarize what I have learned and what I think can be addressed in future work.

2 Existing information for the motion of a surface

The simplest model assumes that the surface is a stretched elastic or metal sheet; using this assumption and small approximations we arrive at the wave equation:

$$c^2 \Delta u(x, y, t) = \frac{\partial^2 u(x, y, t)}{\partial t^2}.$$

Using separation of variables we arrive at the following differential equation:

$$\Delta v(x, y) = -k^2 v(x, y)$$

More complex models which require less assumptions are the Kirchhoff and Mindlin plate theories [5]. However, in this communication we will focus on solutions to the Laplace equation.

3 Problem formulation

Let M be a smooth 2-manifold in R^3 with a boundary and the standard Euclidean metric, and Δ the Laplace operator. A function u is said to be an eigenfunction with eigenvalue λ if $\Delta u = -\lambda u$. With our convention on the sign of Δ , the eigenvalues are non-negative and go to infinity. One of the most fundamental questions about eigenfunctions is to understand the sets where they vanish; these sets are called nodal sets. We will be interested in answering the following question:

Let $\Sigma \subset M$ be a closed hyperspace of M . Does there exist a function u with Σ as its nodal set?

We can break this question into several questions as follows:

- Given Σ , a closed hyper surface of M , does there exist an eigen function with Σ as its nodal set? The answer to this question is not necessarily. Given an arbitrary domain, we can construct infinitely many hypersurfaces such that there doesn't exist an eigen function of the Laplacian with these hypersurfaces as nodal sets.
- Given that the answer to the above question is no, can we find an eigen function that will have as its nodal set an approximation of Σ ? This question might not be well defined since we are not defining precisely what an approximation to an hypersurface is. An acceptable definition for this approximation is as follows: A good approximation of a hyper surface Σ is a hyper surface that contains the same amount of closed curves, the angles of intersection, and the position of the nodal lines relative to some coordinate axis is invariant. what we allow is some stretching or for the slight distortion of the nodal sets, and such that this approximation will be the nodal set of some eigen function on an arbitrary domain (we allow for the change of the domain, as long as it is 2D subset of R^3 , with a boundary, and the standard metric defined). This definitions stems from the observation, that one can not find a domain with two circles as its nodal sets, but one can obtain a domain with topological circles that are a good approximation to circles.
- Is there a systematic way of constructing the approximation to a hypersurface as described above? I don't know the answer to this question. However, in section 5 I summarize some of the properties of eigen functions and their nodal sets; these properties can serve as a guide for the elimination of those hypersurface that can not be constructed.

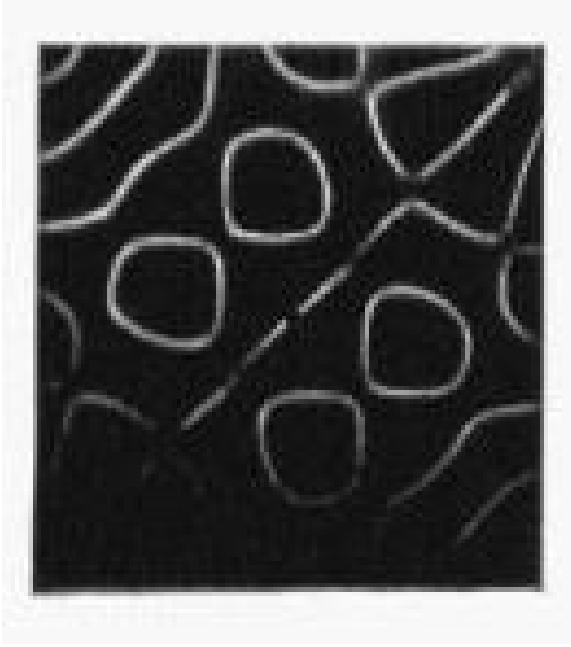


Figure 1: A Chladni pattern featuring four closed curves - topological circles

4 Existing solutions to the Laplace equation

For simple domains with Dirichlet or Newman boundary conditions exact solutions in parabolic or elliptic coordinates are obtained by separation of variables. In fact, for every region $\Omega \subset \mathbb{C}$ and an analytic function $f(z)$ mapping the rectangle R onto Ω with $f(z) = w$ and such that $U(z) = u(w)$ solving the Laplace equation with Dirichlet boundary conditions on Ω is equivalent to solving the following equation on R :

$$\Delta U = -\lambda |dw/dz|^2 U$$

It follows that under a conformal transformation, the laplacian changes by a conformal factor. Moreover, it is shown [4] that the resulting equation can be solved by separation of variables if and only if $f(z)$ is either a quadratic polynomial of z or of the form $a \exp(-z) + b \exp(z)$.

5 On the Eigen functions of the Laplacian and their nodal sets

There are several properties of the nodal sets of the Eigen functions of the Laplacian that I will summarize briefly below:

- The Eigen functions are smooth in the interior of Ω . [4]
- Unique continuation property: any eigen function of the Laplace operator that vanishes on an open set of Ω vanishes on all of Ω . It follows that nodal sets are curves in Ω . [4]
- Where nodal lines cross, they form equal angles. Also, when nodal lines intersect a C^∞ portion of the boundary, they form equal angles. [4]
- Eigen functions can be extended uniquely as a C^∞ function across any C^∞ boundary. It follows that once the eigen function is specified in an open region, it is determined on the entire region, and so the nodal regions are specified. Using this fact one can construct many different geometries of nodal sets that can not be achieved on a 2D surface embedded in R^3 with the standard Euclidean metric.
- In a region possessing an axis of symmetry, the eigen functions can be categorized as symmetric or antisymmetric about the axis. If there are more than one axis of symmetry, the eigen functions can be further partitioned. [4]
- Courant nodal domain theorem:
 - (i) The first eigenfunction $\phi(x)$ corresponding to the smallest eigenvalue cannot have any nodes.
 - (ii) For $n \geq 2$, $\phi(x)$ corresponding to the n th eigenvalue counting multiplicity, divides the domain Ω into at least 2 and at most n pieces.
- The n 'th eigen value of a region Ω is the first eigen value in each of its nodal domains.

6 The physics

Some of the factors that affect the motion of the particles are the elasticity of the plate, friction, the acceleration of the plate, and the size of particles.

The effect of air currents on the motion of the particles is discussed in a paper by Faraday [3], which explains why very fine particles migrate to regions of maximum displacement rather than to the nodal lines resulting in an inverted Chladni pattern. However, air currents are not the only mechanism by which inverted Chladni patterns are formed. It was shown [2] that if the acceleration of the plate is less than the

gravitational acceleration, inverted Chladni patterns will be formed regardless of the size of the particles.

We can perhaps ignore the effect of air currents by assuming that all experiments are in vacuum, unless we want to employ this factor in the formation of patterns, so that we can create more complex patterns using multiple types of particles. Perhaps this can also be used for separation of colors.

It will also be interesting to look at the distribution of the particles across the nodal lines. Since physical particles have a volume and so although in general they will migrate towards the nodal lines, they can't be contained within the nodal lines which are of dimension 1.

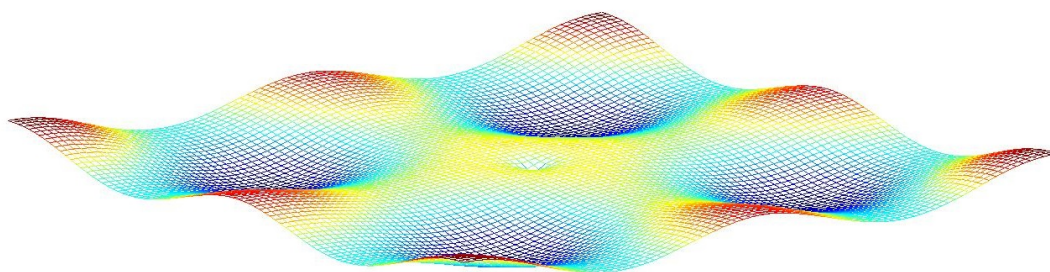


Figure 2: a membrane with free boundary conditions vibrating with a frequency of $\sqrt{18}$

7 Programs

In order to gain an intuition for the problem we constructed a computer program to simulate the motion of a rectangular membrane subjected to various boundary conditions. We employed the finite difference method, along with a forcing function in the form of:

$$f(x, y, t) = \sum_{(x_i, y_i)} \sin(mt) \delta(x - x_i) \delta(y - y_i), (x_i, y_i) \in R$$

where (x_i, y_i) is a discrete close finite subset of R .

By driving the membrane with frequencies which are close in value to the resonant frequencies we were able to simulate the nodal lines. The commented programs are attached in Appendix A.

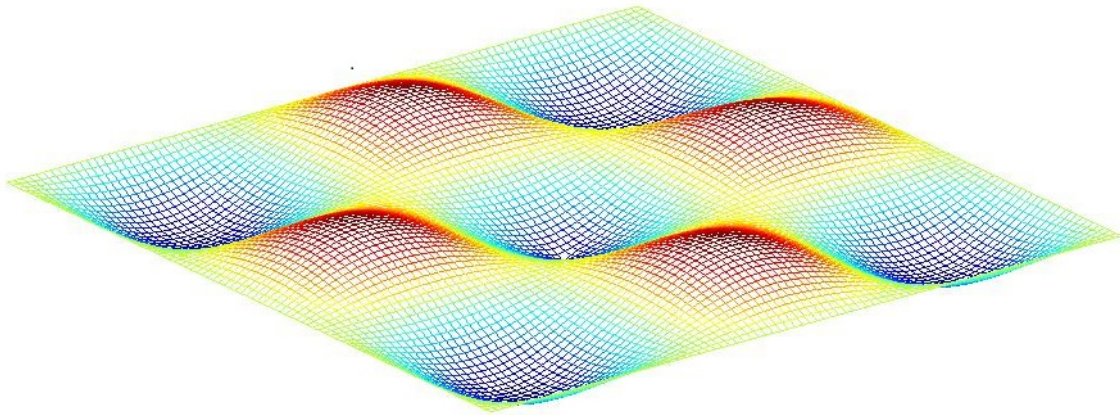


Figure 3: a membrane with fixed boundary conditions vibrating with a frequency of $\sqrt{18}$

8 Future work

Suppose that for a given hypersurface Σ , there doesn't exist any surface such that one of its eigen function has Σ as its nodal set. Can we distort Σ (we would also need to define and measure such distortions) such that a solution does exist? I would like to come up with a way to measure such distortions and study their effect.

One way of investigating the above problem is as follows: by the Riemann mapping theorem there exist an injective biholomorphic map between any proper subset of $\mathbb{C}$ and the unit disk. Under such conformal mappings the Laplacian will change by a conformal factor. We can solve the modified equation on the unit disk using the finite difference method (one can modify the approach in [6] to accomodate the modified equation) and look at the nodal sets of the solutions, and compare them to the existing solution of the Laplacian on the unit disk. This can be done by writing a computer program that will map arbitrary subsets of $\mathbb{C}$ to the unit disk. One can then stretch the domain slightly, break some symmetries etc and look at the affects on the nodal sets. Mapping arbitrary polygons to the disk would be a good place to start. The Schwartz Christoffel mapping is a conformal transformation that maps the interior of a disk to the interior of a polygon. The inverse of such map is needed in the above construction. The MATLAB toolpack available here: <http://www.math.udel.edu/~driscoll/SC/> computes conformal maps to regions bounded by polygons and their inverses.

9 Acknowledgments

I would like to thank Professor Kontorovich and Senia Sheydvasser for their advice, guidance and insight.

Appendix A

MATLAB code for the animation of a vibrating rectangular membrane with free and fixed boundary conditions:

```
clear;

% Parameters to define the wave equation and the range in space and time
% maxt: number of time steps per unit time.
% Tmax: number of time units.
% c: The speed of the wave
% nx,ny: number of spatial steps in the x and y direction
% x,y, time: vectors of dimension nx, ny and max*Tmax respectively.
% u(x,y,time): a matrix storing the displacement as a function of x,y,time.
% lx, ly: points at which the force is applied
% dt: 1/maxt;
% dx: 1/nx;
% dy:1/ny;
% rx: c*dt/dx;
% ry: c*dt/dy;
% m: frequency of the applied force.
% beginA and playA: beginning and end time for the animation.

Tmax =10;
c = 1;
maxt = ceil(88*sqrt(2))*6.;
dt = 1/maxt;
nx = 88;
ny=88.;
u= zeros(nx,ny,maxt*Tmax);
x=linspace(0,1,nx);
y=linspace(0,1,ny);
time=linspace(0,Tmax,maxt*Tmax);
playA=Tmax*maxt;
beginA= 7*maxt;
[X,Y]= ndgrid(x,y);
lx=ceil(nx/2); ly=ceil(ny/2);

dx =1/nx;
dy=1/ny;
rx = c*dt/dx;
ry=c*dt/dy;

kx=3*pi; ky=3*pi;
```



```

m=sqrt(kx^2+ky^2);

% Implementation of the time evolution, using finite difference method

for k=3:maxt*Tmax % Time loop
for i=2:(nx-1) % Space loop
    for j=2:(ny-1)

        if i==lx && j==ly

            u(i,j,k)=sin(m*time(k-1))*dt^2+u(i,j,k-1)*(2-2*(rx^2+ry^2))+
            rx^2*(u(i+1,j,k-1)+u(i-1,j,k-1))+ ry^2*(u(i,j+1,k-1)+
            u(i,j-1,k-1))-u(i,j,k-2);

        else

            u(i,j,k)= u(i,j,k-1)*(2-2*(rx^2+ry^2))+ rx^2*(u(i+1,j,k-1)+
            u(i-1,j,k-1))+ ry^2*(u(i,j+1,k-1)+u(i,j-1,k-1))-u(i,j,k-2);
        end

    end
end

%for a free boundary conditions uncomment the following:
%     for i=(2):(nx-1)
% u(i,1,k) = u(i,2,k);
% u(i,ny,k)=u(i,ny-1,k);
%     end
%
%     for j=(2):(ny-1)
% u(nx,j,k)=u(nx-1,j,k);
% u(1,j,k) = u(2,j,k);
%     end
%
%     u(1,1,k)= u(2,1,k);
%     u(1,ny,k)= u(1,ny-1,k);
%     u(nx,ny,k)= u(nx-1,ny,k);
%     u(nx,1,k)= u(nx-1,1,k);

% for a fixed boundary, uncomment the following:

%     for i=(1):(nx)
% u(i,1,k) = 0;
% u(i,ny,k)=0;
%     end
%
%
```

```

%      for j=(1):(ny)
% u(nx,j,k)=0;
% u(1,j,k) = 0;
%      end

end

xmin=min(x(:)); xmax=max(x(:));
ymin=min(y(:)); ymax=max(y(:));
xmid=(xmin+xmax)/2; ymid=(ymin+ymax)/2;
d=max(xmax-xmin,ymax-ymin)/2; Nt=length(time);
range=[xmid-d,xmid+d,ymid-d,ymid+d,3*min(u(:)),3*max(u(:))];

% In order to record the animation as an Avi file, uncomment
%the commented lines in the rest of the program.

% writerObj = VideoWriter('1-3-4-5.avi');
% open(writerObj);

%Plot positions for successive times

for p=beginA:playA

%  set(gca,'nextplot','replacechildren');
%  set(gcf,'Renderer','zbuffer');
  mesh(X,Y,u(:, :, p)), axis(range);
  xlabel('x axis'), ylabel('y axis')
  zlabel('u axis'), titl=sprintf('MEMBRANE POSITION AT T=%5.2f',time(p));

  title(titl),

  drawnow, shg, pause(.001)
%      frame = getframe;
%      writeVideo(writerObj,frame);

end

%  close(writerObj);

```

References

- [1] J. R. Kuttler, and V. G. Sigillito. *Eigenvalues of the Laplacian in Two Dimensions*. Siam Review, Vol. 26, No. 2, April 1984
- [2] Henk Jan van Gerner, Martin A. van der Hoef, Devaraj van der Meer, and Ko van der Weele. *Inversion of Chladni patterns by tuning the vibrational acceleration*. Phys. Rev. E 82, 012301
- [3] M. Faraday, *On a Peculiar Class of Acoustical Figures; and on Certain Forms Assumed by Groups of Particles upon Vibrating Elastic Surfaces*. Philosophical Transactions of the Royal Society of London Vol. 121 (1831), pp. 299-340
- [4] J. R. Kuttler, V. G. Sigillito, *Eigenvalues of the Laplacian in two dimensions*. Siam Review Vol. 26, No. 2, April 1984
- [5] S.V. Bosakov, *Eigenfrequencies and modified eigenmodes of a rectangular plate with free edges*. Journal of Applied Mathematics and Mechanics
- [6] B. Holman, L. Kunyansky, *A Second-order Finite Difference Scheme For The Wave Equation on a Reduced Polar Grid*.