

On the inverse eigen function problem

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One reason (out of many!) why I find mathematics so fascinating is that it allows one to explore questions in a rigorous manner and on any desired level of abstraction. Some such questions are motivated by physical phenomena, one of which I explored this past summer under the mentorship of Dr. Alex Kontorovich. The topic is discussed in detail below, along with the questions that will be explored in future work.

When a rigid surface is vibrated at certain frequencies, regions of zero displacement are formed. These regions are called nodal lines. Typically the surface is covered with fine particles that migrate to the nodal lines during the vibration, and result in a Chladni pattern. The motivation for this project arose from the following question: given a specific facial pattern, can we find a surface so that when vibrated at some frequency will have this pattern as its nodal domain?

Formally, let M be a connected subset of $\mathbb{R}^2$ with the standard Euclidean metric defined. If we assume that the surface is a stretched elastic sheet, its motion in space and time should satisfy the wave equation: $c^2 \Delta u(x, t) = \frac{\partial^2 u(x, t)}{\partial t^2}$ where $x \in M, t \in [0, 1]$, and $\Delta f = \sum_{i=1}^3 \frac{\partial f}{\partial x_i}$. We will assume that we can write $u(x, t) = f(x)g(t)$, where f, g are of class C^∞ . After separation of variables we obtain: $\Delta f = -\lambda f$.

The function f is said to be an eigenfunction of the Laplace operator with eigenvalue λ . One of the most fundamental questions about eigenfunctions is to understand the sets on which they vanish, the nodal sets. Typically one is given the value of a function on the boundary of M , and using analytic and approximation techniques one obtains the eigenfunctions which satisfy the boundary conditions. Once one obtains the eigen functions, one can study their associated nodal sets.

What we are interested in is the reverse problem. Let $\Sigma \subset M$ be a closed hyperspace of M . Supposing that the boundary conditions are not specified, and can be adjusted at will. Does there exist a function f with Σ as its nodal set? The answer to this question is not necessarily. For example, let M be a connected subset of $\mathbb{R}^2$, and Σ two non intersecting circles of radius 1, say. There exists a unique function which vanishes exactly on the first circle, namely the J_0 Bessel function. This function is radially symmetric, and hence the value of the function is determined and can not vanish on the second circle.

In general, Eigen functions can be extended uniquely as a C^∞ function across any C^∞ boundary. It follows that once the eigen function is specified in an open subset of $\mathbb{R}^2$, it is determined on the entire domain, and so the nodal sets are specified. Using

this fact one can construct many different geometries of nodal sets that can not be achieved on a 2D surface with the standard Euclidean metric.

Given that the answer to the previous question is no, we can follow with the following questions:

- If we vary the domain and the value of the function on its boundary, but we fix the metric (Euclidean), is there a way of finding/characterizing all the allowed hyper surfaces? Can we answer this question for $2D$ smooth manifolds $\in \mathbb{R}^3$?
- Given Σ , a hyper surface that is not a nodal set of any eigen function, can we find an eigen function that will have as its nodal set an approximation of Σ ?

In order to answer the first question, one can study the properties of eigen functions and their associated nodal sets. For example, we can study the angle of intersection of the nodal sets in relation to the domain and boundary conditions. We can study the nodal sets of domains possessing an axis of symmetry. Some of these properties are discussed in [1].

In order to answer the second question we need to define what a good approximation of Σ is. Intuitively, what we allow for is slight stretching, bending, and distortion. These descriptions stem from the observation that one can not find a domain with two circles as its nodal sets, but one can obtain a domain with topological circles that as far as the eye can tell are a good approximation to circles. We allow for the change of the domain, as long as it is a 2D manifold of class C^r , $r > 0$, embedded in R^3 , with a boundary. In order to make this precise, we need to define a metric that will measure the difference between Σ and its approximation. We should require that such an approximation of Σ should contain the same amount of path connected components. Moreover, If the nodal sets intersect, we would like to be able to measure the variation of the intersection angle. In defining such metric we would need to account for all pathological examples, which is a good exercise and something i really enjoy!

It should be noted that this problem can be discussed in higher dimensions as well, and with different geometries defined. In fact, if we fix M but allow the geometry to vary, then it can be shown that a solution exists. Formally: if M is a compact n manifold, and Σ is a closed separating hypersurface of M , there is a Riemannian metric on M such that the nodal set of its first nontrivial eigenfunction is Σ . [2]

- [1] J. R. Kuttler, and V. G. Sigillito. *Eigenvalues of the Laplacian in Two Dimensions*. Siam Review, Vol. 26, No. 2, April 1984
- [2] Alberto Enciso, Daniel Peralta-Salas, *Eigenfunctions with prescribed nodal sets* arXiv:1404.1039 [math.DG]