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AUTOMORPHISM GROUPS OF PLANAR GRAPHS. I

László BABAI
Eötvös Loránd University, Budapest, Hungary

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Abstract. The existence of groups which are not isomorphic with the group of automorphisms of any planar graph is proved. Subsequent results are mentioned.

§ 1. Introduction

By graphs and groups we mean in this paper finite ones. A graph will be called *simple* if it is undirected and has neither loops nor multiple edges. Frucht proved in [2] that for an arbitrary group G there exist (infinitely many non-isomorphic) simple graphs X the automorphism groups $A(X)$ of which are isomorphic with G.

Let us say that the group G *allows* a class K of graphs (or more general relational systems), if there exists an $X \in K$ such that $A(X) \cong G$. Sabidussi proved in [6] that the groups of order > 1 allow a number of important classes of simple, fixed-point-free graphs, e.g., the n-connected graphs ($n \geq 1$ a given integer), the regular graphs of given degree ≥ 3, etc. Generalizations and further properties of graphs, allowed by all finite groups can be found in [1,4].

A very important class is the class of the planar graphs. Turán raised the following question in 1969: Given a finite group G, does there exist a planar graph X such that $A(X) \cong G$? In this paper we answer this question in the negative (see Theorem 1). The study of this question seems to be rather difficult, in particular the basic problem: Which groups allow the class of planar graphs? It is easy to show (Theorem 3) that the finite abelian groups allow the class of (finite, simple, connected) planar graphs. In later papers we determine a rather large class of groups which do not allow the class of planar graphs.

Further related problems are the following: Which groups allow the class of connected (2-connected, 3-connected, connected and fixed-point-free) planar graphs (or, more generally, graphs which can be drawn on a given surface). We will deal with these questions in later papers.

As a technical device to be used in this and in later papers we shall define the *shadow* $\widetilde{X}$ of a relational system X with unary and binary relations; $\widetilde{X}$ is a simple graph. The proofs of the negative results will be formulated for the class of relational systems with unary and binary relations the shadows of which are planar graphs (or connected, 2-connected planar graphs, etc.). But this is not an essential generalization, as a simple construction shows (Theorem 2).

§ 2. Definitions and notation

A *relational system* X is a pair $(V, \mathcal{R})$ where $V = V(X)$ is a non-void set, the set of vertices of X, $\mathcal{R} = \mathcal{R}(X) = \{R_\alpha \mid \alpha \in A\}$ is a family of relations defined on V. If $\mathcal{R}$ consists only of unary and binary relations (subsets of V and of $V \times V$, resp.) then $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_1$ contains the unary and $\mathcal{R}_2$ the binary relations. If $\mathcal{R} = \mathcal{R}_2 = \{R\}$, then X is a graph; $E(X) = R$ is the set of edges of X. X is *undirected* if R is symmetric. In the general case if $(x, y) \in R_\alpha \in \mathcal{R}_2$, we can say that there is a "coloured edge" of "colour" α in X directed from x to y, where $x, y \in V(X)$. We will deal only with finite X, i.e., V and $\mathcal{R}$ are finite.

The *automorphisms* (*endomorphisms*, resp.) of X are those permutations (transformations, resp.) of $V(X)$ which preserve the relations R_α; they form the group $A(X)$ (monoid $C(X)$, resp.). We shall say that X is *fixed-point-free* if the permutation group $A(X)$ is fixed-point-free. If p, q are permutations acting on the set S, we use the convention $(pq)(x) = p(q(x))$ for every $x \in S$. We denote by $P|T$ the restriction of the permutation group P to the set T if this restriction is defined, i.e., if T is invariant under all elements of P.

Let $X = (V, \mathcal{R})$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. We define a simple graph $\widetilde{X}$, called the *shadow* of X, as follows:

$$V(\widetilde{X}) = V(X),$$

$$E(\widetilde{X}) = \{(x, y), (y, x) \mid x, y \in V(X),\ x \neq y, \text{ there is an } R_\alpha \in \mathcal{R}_2 \text{ such that } (x, y) \in R_\alpha\}.$$

Clearly $A(\widetilde{X}) \supseteq A(X)$; if X is a graph, then $\widetilde{X}$ is planar ($\widetilde{X}$ can be drawn on a given surface) if and only if the same is true for X. If X is simple, then $\widetilde{X} = X$.

Let G be a group, $H \subseteq G$. The "Cayley colour graph" $X_{G,H}$ is the following relational system with binary relations:

$$V(X_{G,H}) = G \; ,$$

$$\mathcal{R}(X_{G,H}) = \mathcal{R}_2(X_{G,H}) = \{R_h \mid h \in H\} \; ,$$

where

$$R_h = \{(g, gh) \mid g \in G\} \; .$$

The (left) regular representation P_G of G consists of the permutations $\pi_g, g \in G$, defined by $\pi_g(x) = gx$, where $x \in G$.

Some trivial facts: $P_G \cong G$; for any $H \subseteq G$, $A(X_{G,H}) \supseteq P_G$, consequently $A(\widetilde{X}_{G,H}) \supseteq P_G$. On the other hand, if Y is a simple graph and $A(Y) \supseteq P_G$, then $Y = \widetilde{X}_{G,H}$ for a suitable $H \subseteq G$ (cf. [7]). $\widetilde{X}_{G,H}$ is connected if and only if H is a generating system of G.

Let X and Y be simple graphs with

$$V(X) = \{x_0, x_1, ..., x_n\} \; , \qquad V(Y) = \{y_1, ..., y_n\} \; .$$

The mapping $\psi : V(X) \to V(Y)$ defined by

$$\psi(x_i) = \begin{cases} y_1 & \text{if } i = 0 \; , \\ y_i & \text{if } i = 1, ..., n \end{cases}$$

is called *contraction* of the edge (x_0, x_1) if

$$(x_0, x_1) \in E(X)$$

and

$$(y_i, y_j) \in E(Y) \Longleftrightarrow (x_i, x_j) \in E(X), \; 2 \le i < j \le n \; ;$$

$$(y_1, y_j) \in E(Y) \Longleftrightarrow (x_1, x_j) \in E(X) \qquad \text{or}$$

$$(x_0, x_j) \in E(X), \; 2 \le j \le n \; .$$

In this case we denote $Y = \psi(X)$.

Let $\phi = \psi_k \ldots \psi_1$, where the ψ_i's are contractions of edges. Then ϕ is called a *connected-homomorphism*. (ϕ is not a homomorphism.) Let $Z = \phi(X)$. Then clearly Z is also a simple graph, and if X is planar, then so is Z. More generally, if X can be drawn on a given surface, then Z can be drawn on the same surface.

§ 3. The main theorem

Theorem 1. *There are infinitely many finite groups which are not isomorphic with $A(X)$ for any planar graph X.*

In this paper we prove that the generalized quaternion groups Q^n have this property. Q^n, $n \geq 3$, is defined by

$$Q^n = \langle \alpha, \beta \mid \alpha^{2^{n-1}} = 1, \;\; \beta\alpha\beta^{-1} = \alpha^{-1}, \;\; \beta^2 = \alpha^{2^{n-2}} \rangle .$$

$|Q^n| = 2^n$ and $\gamma = \beta^2$ is the only element of Q^n of order 2. Hence, if $x \in Q^n$, $x \neq 1$, then γ is a power of x (since $2 \mid o(x)$, and $o(x^{\frac{1}{2}o(x)}) = 2$).

Lemma. *If H is a generating system of the generalized quaternion group Q^n, then the shadow of the Cayley colour graph $X_{Q^n,H}$ is not a planar graph.*

Proof. Since H generates a non-commutative group, it contains two non-commuting elements a and b, i.e., $ab \neq ba$. Let $Y = \widetilde{X}_{Q^n,\{a,b\}}$. So $Y \subseteq \widetilde{X}_{Q^n,H}$.

Clearly $o(a) \neq 2$, $o(b) \neq 2$. Hence xa, xa^{-1}, xb, xb^{-1} are four different elements of Q^n ($x \in Q^n$). That is, the graph Y is regular of degree 4.

Y does not contain any triangle. To prove this statement we have to consider all products of three factors consisting of elements a, a^{-1}, b, b^{-1}, and to prove that these products are different from the unit element of Q^n. But this is obvious since any of the equalities

$$a^{\pm 1}a^{\pm 1}b^{\pm 1} = 1 \,, \quad a^{\pm 1}b^{\pm 1}a^{\pm 1} = 1 \,, \quad b^{\pm 1}a^{\pm 1}a^{\pm 1} = 1$$

would imply that b is a power of a which would contradict the condition that $ab \neq ba$.

Finally,

$$a^{\pm 1}a^{\pm 1}a^{\pm 1} = 1$$

would imply $a = 1$, since $(o(a), 3) = 1$, which is also a contradiction. We obtain the remaining cases by interchanging the roles of a and b.

Assume that Y is drawn in the plane. Let v be the number of vertices of Y, e the number of edges and f the number of faces. Since the vertices have valency 4, $e = 2v$. Since there is no triangle in Y, $e \geq 2f$ (the boundary of each face consists of at least 4 edges and an edge belongs to the boundaries of 2 faces). Consequently, $e \geq v + f$, but on the other hand $v + f > e$. This contradiction proves the lemma.

Proof of Theorem 1. Let X be a relational system with unary and binary relations. Let $|V(X)| = k$. We prove by induction on k that

(1) if $A(X) \cong Q^n$, then the shadow $\widetilde{X}$ of X is not a planar graph.

Let $A(X) = P \cong Q^n$. Let p be the only involution in P (the permutation corresponding to $\gamma \in Q^n$). Since p is not the identity permutation ϵ of $V(X)$ there is an $x \in V(X)$ with $p(x) \neq x$. For any $q \in P$, $q(x) = x$ implies $q = \epsilon$, since otherwise p is a power of q.

For any vertex $a \in V(X)$ we will denote by T_a the orbit of P containing a.

We now consider the vertices y of $\widetilde{X}$ which are adjacent to x. There are two cases to distinguish.

Case (a). Assume that $y \in V(X) \setminus T_x$ and $(x, y) \in E(\widetilde{X})$ imply $p(y) = y$.

Let K be the full subgraph of $\widetilde{X}$ on T_x, and $S = P | T_x$. Clearly $S \cong P \cong Q^n$; and if we identify the elements of T_x with the elements of Q^n by the one-to-one correspondence $q(x) \to q$, we obtain that S is the (left) regular representation of Q^n. Clearly $S \subseteq A(K)$. Hence $K = \widetilde{X}_{Q^n, H}$ where H is a suitable subset of Q^n.

If H is a generating system of Q^n then K is not a planar graph — this was the assertion of the lemma; hence (because of $K \subseteq \widetilde{X}$) we are finished.

If H does not generate Q^n, then neither does $H' = H \cup \{p\}$. Hence the graph $K' = \widetilde{X}_{Q^n, H'}$ is disconnected. Let L be a connected component of K' which does not contain the vertex x. L is invariant under p since $p \in H'$. We define a permutation q of $V(X)$ by

$$q(u) = u \qquad \text{if} \quad u \notin V(L)\,,$$

$$q(u) = p(u) \quad \text{if} \quad u \in V(L) .$$

This is indeed a permutation of $V(X)$, since L is invariant under p. $q \in A(X)$ because if $R \in \mathcal{R}_2(X)$, $u \in V(L)$, $v \notin V(L)$ (all the other cases are trivial), then $(u,v) \in R$ as well as $(v,u) \in R$ imply $v \notin T_x$ and therefore $p(v) = v$ (this is the assumption underlying part (a) of this proof); hence $q(v) = p(v)$, $q(u) = p(u)$. If $(u,v) \in R$, then $(p(u), p(v)) \in R$, and this implies $(q(u), q(v)) \in R$. Similarly, if $(v,u) \in R$, then $(p(v), p(u)) \in R$, and this implies $(q(v), q(u)) \in R$. Finally, we have (i) $q(x) = x$ since $x \notin V(L)$, and (ii) $q \neq \epsilon$ since none of the vertices of L remains fixed under q. The latter two statements give a contradiction since $p(x) \neq x$ implies $q(x) \neq x$ for each $q \in A(X)$, $q \neq \epsilon$. This contradiction proves that H in any case generates Q^n; this completes the proof of case (a), and at the same time starts the induction.

Case (b). Assume that there is a vertex $y \in V(X)$ with $p(y) \neq y$ and $(x,y) \in E(\widetilde{X})$, and which does not belong to the orbit of ^{P} containing x.

Clearly $q(x) = r(x) \iff q(y) = r(y) \iff q = r$, where $q, r \in P$.

Let us denote the elements of the set T_z by z_q, $q \in P$. We define a relational system X_1 containing only unary and binary relations on the set

$$V(X_1) = V(X) \cup T_z \setminus T_x \setminus T_y \ .$$

We first define some unary and binary relations on $V(X_1)$. For $R \in \mathcal{R}_1(X)$, let

$$R' = R \cap V(X_1) .$$

For $R \in \mathcal{R}_2(X)$, let

$$\begin{aligned} R' = {} & \{(u,v) \mid (u,v) \in R,\ u,v \in V(X) \cap V(X_1)\} \\ & \cup \{(u,z_q) \mid (u, q(x)) \in R,\ u \in V(X) \cap V(X_1)\} \\ & \cup \{(z_q, v) \mid (q(x), v) \in R,\ v \in V(X) \cap V(X_1)\} \\ & \cup \{(z_q, z_r) \mid (q(x), r(x)) \in R\} . \end{aligned}$$

$$R'' = \{(u, z_q) \mid (u, q(y)) \in R,\ u \in V(X) \cap V(X_1)\}$$

$$\cup \{(z_q, v) \mid (q(y), v) \in R,\ v \in V(X) \cap V(X_1)\}$$

$$\cup \{(z_q, z_r) \mid (q(y), r(y)) \in R\}\,.$$

$$R''' = \{(z_q, z_r) \mid (q(x), r(y)) \in R\}\,,$$

$$R^{\mathrm{IV}} = \{(z_q, z_r) \mid (q(y), r(x)) \in R\}\,.$$

Finally let

$$\mathcal{R}_1(X_1) = \{T_z\} \cup \{R' \mid R \in \mathcal{R}_1(X)\}\,,$$

$$\mathcal{R}_2(X_1) = \{R', R'', R''', R^{\mathrm{IV}} \mid R \in \mathcal{R}_2(X)\}\,.$$

For $q \in P$, let the permutation q' of $V(X_1)$ be defined as follows

$$q'(u) = q(u) \quad \text{if} \quad u \in V(X) \cap V(X_1)\,,$$

$$q'(z_r) = z_{qr}\,.$$

It is clear that in this way we obtain indeed an automorphism q' of X_1; the correspondence $q \to q'$, where $q \in P$, is one-to-one and $(qr)' = q'r'$. That is, the set of permutations

$$P' = \{q' \mid q \in P\}$$

is a subgroup of $A(X_1)$ and $P' \cong P \cong Q^n$.

By the definition of X_1, one can easily prove that every automorphism of X_1 is of the form q', where q is a suitable element of P. For if $t \in A(X_1)$, let us define the permutation q of $V(X)$ as follows:

$$q(u) = t(u) \quad \text{if} \quad u \in V(X) \cap V(X_1)\,,$$

$$q(u) = r(x) \quad \text{if} \quad u = s(x) \quad \text{and} \quad t(z_s) = z_r\,,$$

$$q(u) = r(y) \quad \text{if} \quad u = s(y) \quad \text{and} \quad t(z_s) = z_r, \quad \text{where} \quad r, s \in P\,.$$

In this way we really defined a permutation of $V(X)$. T_z being invariant under t because of $T_z \in \mathcal{R}_1(X_1)$. The fact that $q \in A(X)$ is an immediate consequence of the definition of $\mathcal{R}_1(X_1)$ and $\mathcal{R}_2(X_1)$; and clearly $t = q'$. Consequently $A(X_1) = P'$.

Let the mapping $\phi : V(X) \to V(X_1)$ be defined as follows:

$$\phi(u) = u \quad \text{if} \quad u \in V(X) \cap V(X_1) ,$$

$$\phi(u) = z_q \quad \text{if} \quad u = q(x) \quad \text{or} \quad u = q(y) .$$

By construction, ϕ is a connected-homomorphism from $\widetilde{X}$ onto $\widetilde{X}_1$.

$|V(X_1)| < |V(X)|$, hence by the induction hypothesis, $\widetilde{X}_1$ is not a planar graph. In consequence of the existence of the connected-homomorphism ϕ, the graph $\widetilde{X}$ is likewise non-planar.

§ 4. Constructions

The following theorem shows that (1) is not much more general than the original Theorem 1.

Theorem 2. *Let X be a finite relational system containing only unary and binary relations. Assume that $\widetilde{X}$ is a planar graph. Then there exists a simple planar graph Y such that $A(Y) \cong A(X)$. Moreover, if $\widetilde{X}$ is connected (2-connected, fixed-point-free) then Y may be assumed to have the same property.*

Proof. First we get rid of unary relations and of vertices which do not belong to any pair $(x,y) \in R$ for any $R \in \mathcal{R}_2(X)$.

Let X' be the relational system containing only binary relations defined as follows:

$$V(X') = V(X) ;$$

$$\mathcal{R}(X') = \mathcal{R}_2(X') = \mathcal{R}_2(X) \cup \{S_R \mid R \in \mathcal{R}_1(X)\} \cup \{I\} ,$$

where

$$S_R = \{(x,x) \mid x \in R\} ;$$

$I = \{(x,x) \mid (x,y) \notin R \text{ and } (y,x) \notin R \text{ for any } y \in V(X),\ R \in \mathcal{R}_2(X)\}$.

Clearly $\widetilde{X}' = \widetilde{X}$ and $A(X') = A(X)$.

In what follows, we apply a well-known method known under the name of *šip-construction* (cf. [3, 5]).

Let us write $\mathcal{R}(X') = \{R_i \mid i = 2, \ldots, n\}$. For each $i = 2, \ldots, n$, we define a directed graph D_i by

$$V(D_i) = \{(k,i) : 0 \leq k \leq 2i\},$$

$$E(D_i) = \{((k,i),(k+1,i)) \mid 0 \leq k \leq i-1\}$$

$$\cup \{((k+1,i),(k,i)) \mid i \leq k \leq 2i-1\}$$

$$\cup \{((0,i),(2i,i))\},$$

and we put

$$a_i = (0,i) \in V(D_i), \qquad b_i = (i,i) \in V(D_i).$$

Next, we define a directed graph X'' as follows:

$$V(X'') = (\bigcup_{i=2}^{n} V(D_i) \times R_i)/\mathrm{E},$$

where E is the equivalence relation on $\bigcup_{i=2}^{n} V(D_i) \times R_i$ generated as follows:

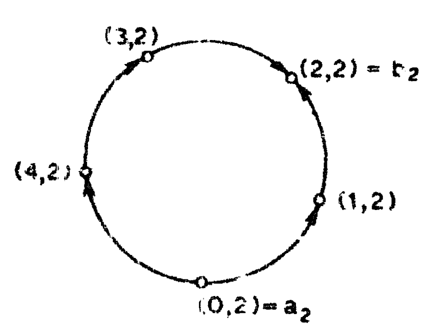

Fig. 1. The graph D_2.

(1) if $d \neq a_i, b_i$ for every $i = 2, ..., n$, then
$(d,s) \mathrm{E} (d',s') \Longleftrightarrow (d,s) = (d',s')$,

(2) $(a_i,(x_1,x_2)) \mathrm{E} (a_j,(x'_1,x'_2)) \Longleftrightarrow x_1 = x'_1$,

(3) $(b_i,(x_1,x_2)) \mathrm{E} (a_j,(x'_1,x'_2)) \Longleftrightarrow x_2 = x'_1$,

(4) $(b_i,(x_1,x_2)) \mathrm{E} (b_j,(x'_1,x'_2)) \Longleftrightarrow x_2 = x'_2$.

We denote by $d * s$ the equivalence class containing (d,s). Let

$$E(X'') = \{(d_1 * s_1, d_2 * s_2) \quad \text{there are } d'_1, d'_2, s' \text{ such that}$$

$$d_1 * s_1 = d'_1 * s', d_2 * s_2 = d'_2 * s', (d'_1, d'_2) \in \textstyle\bigcup_{i=2}^{n} E(D_i)\}$$

In other words, the graph X'' is obtained by replacing every couple $(x_1,x_2) \in R_i$ ("coloured edge") in X' by a copy of D_i in such a way that a_i corresponds to x_1 and b_i to x_2. Clearly $E(X'')$ is irreflexive. The proof that $A(X') \cong A(X'')$ is analogous to that of Theorem 1 in [3]:

Let $f \in A(X')$. Define the mapping $\Phi(f)$ from $V(X'')$ onto itself by

$$\Phi(f)(d * (x_1,x_2)) = d * (f(x_1), f(x_2)) .$$

This definition does not depend on the choice of the representative of the class $d * (x_1,x_2)$. Obviously $\Phi(f) \in A(X'')$ and Φ defines a homomorphism of $A(X')$ into $A(X'')$. Let $f,g \in A(X')$, $f \neq g$. Hence $f(x) \neq g(x)$ for some $x \in V(X')$. As a consequence of the construction of X', there is a $z \in V(X')$ such that either $(x,z) \in R_i$ or $(z,x) \in R_i$ for some i, $2 \leq i \leq n$. Consider a vertex $d \in V(D_i) \setminus \{a_i, b_i\}$ (this is possible since by definition $|V(D_i)| > 3$). First let $(x,z) \in R_i$. We have

$$\Phi(f)(d * (x,z)) = d * (f(x), f(z))$$

$$\neq d * (g(x), g(z)) = \Phi(g)(d * (x,z)) .$$

Similarly for $(z,x) \in R_i$. Hence Φ is a monomorphism.

Let $g \in A(X'')$. The proof that $A(X') \cong A(X'')$ will be finished if we show that $g = \Phi(f)$ for some $f \in A(X')$. As the elements of the set

$$V = \{d * (x_1,x_2) \mid d \in V(D_i) \setminus \{a_i, b_i\}, (x_1,x_2) \in R_i , i = 2, ..., n\}$$

are the only vertices of X'' which are end vertices of exactly one edge,

V is invariant under all automorphisms of X'' The only chains in X'' of length i and $i+1$, whose inner vertices are elements of V, connect $a_i * (x_1, x_2)$ to $b_i * (x_1, x_2)$, where $(x_1, x_2) \in R_i$.

Moreover, the class $a_i * (x_1, x_2)$ depends only on x_1 but not on i and x_2; similarly, $b_i * (x_1, x_2)$ depends only on x_2. Hence there is a mapping f defined on $V(X')$ such that

$$g(a_i * (x_1, x_2)) = a_i * (f(x_1), z) ,$$

$$g(b_i * (x_1, x_2)) = b_i * (u, f(x_2)) ,$$

where $(x_1, x_2) \in R_i$, $i = 2, \ldots, n$, and $z, u \in V(X')$. It follows easily that $f \in A(X')$, and $g = \Phi(f)$. Thus, $A(X'') \cong A(X)$.

Clearly X'' is fixed-point-free whenever X is so. The graph $\widetilde{X}''$ is obviously planar (connected) if and only if $\widetilde{X}'$ is so. If $\widetilde{X}'$ is 2-connected and there are no loops in X' (i.e., $(x, x) \notin R_i$ for every $x \in V(X')$, $i = 2, \ldots, n$), then $\widetilde{X}''$ is also 2-connected.

If $\widetilde{X}$ is 2-connected, we can avoid the loops of X'; instead of X' we consider the following relational system X_o:

$$V(X_o) = V(X) = V(X'),$$

$$\mathcal{R}(X_o) = \mathcal{R}_2(X_o) = \{R_i' \mid 2 \leq i \leq n\} \cup \{S_{ij}^k \mid 2 \leq i, j \leq n,\ k = 1, 2\},$$

where

$$R_i' = R_i - \{(x, x) \mid x \in V(X)\} ;$$

$$S_{ij}^1 = R_i \cap \{(x_1, x_2) \mid (x_1, x_1) \in R_j,\ x_1 \neq x_2\} ;$$

$$S_{ij}^2 = R_i \cap \{(x_1, x_2) \mid (x_2, x_2) \in R_j \quad x_1 \neq x_2\} .$$

If there is at most one isolated point in X, then obviously $A(X) = A(X_o)$, $\widetilde{X} = \widetilde{X}_o$, and X_o contains neither loops nor unary relations. If we use X_o in place of X', the resulting graph $\widetilde{X}''$ is 2-connected whenever $\widetilde{X}$ is so.

To complete the proof of Theorem 2 we apply the so-called Šip-product (for a definition see [3, 5]). Let

$$Y = W * X'' ,$$

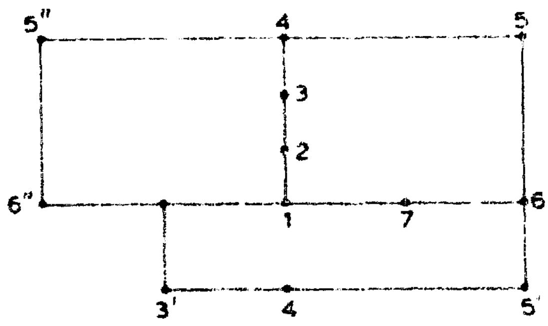

Fig. 2.

where $*$ denotes Šip-product; W denotes the undirected Šip-indikator denoted by $(T, Z, 5', 5'')$ in both [3] and [5] (see fig. 2). That means, we obtain Y by replacing every edge in X'' by a copy of W in such a way that $5'$ is the initial and $5''$ the terminal vertex. W is a planar graph, and the vertices $5'$ and $5''$ belong to the boundary of the same face; hence if X'' is planar, so also is Y. Since X'' does not contain isolated points and $E(X'')$ is irreflexive, we obtain by Theorem 1 and Lemma 2 of [3] that

$$C(Y) = C(W * X'') \cong C(X'') .$$

From this follows $A(Y) \cong A(X'')$.

Y is a simple graph. If X'' has no fixed points, neither does Y. If $\widetilde{X}''$ is connected (2-connected), so also is Y.

Remark 1. The same construction results in a simple graph Y which is embeddable in a given surface whenever $\widetilde{X}$ is so. This is obvious if we consider the above proof.

Theorem 3. *Given any finite abelian group G there is a simple planar graph X such that $A(X) \cong G$. Furthermore, X can be so constructed as to have one of the following properties:*

(a) *X is fixed-point-free if $|G| > 1$;*

(b) *X is connected.*

Remark 2. In a later paper we shall prove that there are abelian groups which allow neither the 2-connected planar graphs nor the planar graphs having simultaneously properties (a) and (b).

Proof of Theorem 3. Let $G = Z(n_1) \times \ldots \times Z(n_k)$ where $Z(n)$ denotes the cyclic group of order n. Let X_1 and X_2 be the relational systems containing only binary relations defined as follows:

$$V(X_1) = \{(i,j) \mid 1 \le i \le k,\ 1 \le j \le n_i\}\ ,$$

$$\mathcal{R}(X_1) = \mathcal{R}_2(X_1) = \{R_i \mid 1 \le i \le k\}$$

where

$$R_i = \{((i,j),(i,j+1)) \mid 1 \le j \le n_i - 1\} \cup \{((i,n_i),(i,1))\}\ .$$

Obviously $A(X_1) \cong G$, and $\tilde{X}_1$ is planar. Moreover, X_1 is fixed-point-free if $n_i \neq 1$ for $i = 1, \ldots, k$ (which we can assume whenever $|G| > 1$).

Now let $v \notin V(X_1)$ and put

$$V(X_2) = V(X_1) \cup \{v\}\ ,$$

$$\mathcal{R}(X_2) = \mathcal{R}_2(X_2) = \mathcal{R}_2(X_1) \cup \{R\}\ ,$$

where

$$R = \{(v,x) \mid x \in V(X_1)\}\ .$$

Obviously $A(X_2) \cong G$, and $\tilde{X}_2$ is connected and planar. Application of Theorem 2 to X_1 and X_2, respectively, completes the proof.

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