Math Modeling of Crowd Dynamics by Benedetto Piccoli

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Abstract. During the holiday shopping season, malls seem to be as crowded as busy city streets. It's a pedestrian traffic jam from store to store as people try to navigate the pathways that will lead them to the perfect holiday gift, and maybe even a bargain. Trying to get around or through droves of people isn't just a science perfected by savvy shoppers. This tutorial brings together two disciplines when analyzing crowd dynamics: psychology, which studies the cognitive processes behind the action of walking; and mathematics, which attempts to quantify the laws that govern the way crowds of people move or interact. Topics will include:
The phenomenology of crowd dynamics: self-organization, patterns and cognitive processes. Experiments with crowds: what to measure, how to measure, experimental settings. Modeling crowd dynamics: choice of the scale, ODE and PDE models, new measure theory approach. Measure-theory multi-scale models: math behind, properties of the model, numerics and simulations.

1 Introduction

This research is about crowd dynamics and it is very interdisciplinary. It is a field where you find a lot of approaches and We think you need a lot of approaches to understand crowds. The first part of the lecture will be formula free, as our book is.

Let's start by taking a very broad point of view. There are so many different ways of interpreting the word "crowd dynamics". There are studies that are related to physiology, and there are biological studies as well.

Fig. 1. Intelligent groups

The topic can be divided into several *Intelligent groups* including:

- Vehicular traffic,
- Crowd dynamics,
- Animal groups,
- Networked robots.

We are interested in what happens when there is a large crowd moving in a city. Similarly with animal groups. The networked robots are something that you would like to apply the study for. If you understand how the other systems work, you can transfer this knowledge to robots.

The main difference between crowd dynamics and animal groups is the environment. In our society there are many cultural rules. For example, depending on the country, you should stay on the corresponding side of the road. (*Left side, right side, whatever side you* *are on.*) For animals there are much simplified rules for their interactions because they usually do not appear in such a structured environment as humans do.

There is a difference in how would a single pedestrian behave when navigating through cities, buildings,... and what would a group of pedestrians do. New phenomenons appear in groups.

1.1 Agent Properties

Studying crowds is different from studying fluids because each single agent can do a lot. He is "intelligent" in comparison to a drop of water.

Agents behave differently than fluids do. — Individuals can use energy. For example when you are driving, you can accelerate or brake. There is a parallel with giving and taking energy from the crowd. You can define an energy reserve for each agent.

Another difference is, that agents can have different targets which is not possible in case of fluid. That is one of the major problems when you try to simulate crowd dynamics using fluid models.

1.2 How to avoid collisions

How people avoid collisions on a pathway? It is very different among societies but some phenomenons do not change at all. Imagine people standing in a queue to get on a train. They form a shape that is called the *arch*, instead of forming a line. It looks like arches around the door (or any kind of a narrow passageway).

1.3 Models

A natural approach for modeling crowds is to use existing fluid dynamics models. While such simple models work very well for vehicular traffic, this is not the case with crowds.

Vehicular traffic is an one-dimensional problem because of the existence of the lanes. There is a line change which makes it into a slightly more dimensional problem, but because of dominant motion in one direction it remains effectively one-dimensional.

On the other hand, crowd dynamics is a full two-dimensional problem. While it is still possible to use existing fluid dynamics models, they don't usually work very well. One of the biggest limitation of fluid model is that it can be used only in situation with one target. In crowd dynamics are often studied cases with more groups of people with different targets.

1.4 Phenomena

The first thing you can do when you study a system is to *observe* it. Even though there are a lot of differences between countries and societies there are a lot of similarities as well.

Vehicular traffic has a density threshold beyond which congestion appears quite consistently. For low density of traffic the flow increases almost linearly. After reaching some density threshold, the behaviour is not completely the same but always the flow decreases. Finding the density threshold can be used in management of real traffic situations because we can maximize the flow.

We would like to summarize the most common and interesting Phenomenology of crowds:

- Lane formation in unidirectional flows.
- Walking couples, clusters and V's.
- Capacity drops at bottlenecks.
- Clogging and arches in stampedes and panic situations.
- Slower-is-faster effect and Braess' paradox for obstacles.
- Zipper configuration at bottlenecks.
- Coexisting lanes formation in opposite flows.
- Circular flows at cross intersections.

Fig. 2. Phenomenology of Crowds

- Intermittent flows (or traffic light effect) at shared bottlenecks.

Arches

The arches are a dangerous configuration and they are so inefficient. There are many examples when this lead to people getting killed. The problem with arches is that in emergency situation tries more people to pass though a bottleneck and they get stuck.

Inverted V

When 3 or more pedestrians walk next to each other, the middle one keeps half-a-step back because that way the whole group can communicate. In contrast to geese or cranes flying in V-shapes which is not well understood, but believed to be efficient at least from the aerodynamics point of view.

Lanes

Lanes form on the street crossing when two groups of people are walking towards each other. There are models which shows how the lanes are changing over time until all lanes are thick enough and the formation get into stable position.

Breess' paradox

Breess' paradox in crowd dynamics claims that placing additional obstacles (next to an exit) increases the flow of the crowd rapidly. It is also unintuitive why this is true because it should be more difficult with less space, should it not? We know it works that way because it has been observed many times with exactly the same results. This paradox is used in many cities – for example additional obstacles are placed in underground in NYC.

Another paradox is why people stop outside of the building on fire, which it is the reason to designate "meeting places", so that people will move further from the building and provide space for others.

2 Examples of crowd behavior

How people behave when they exit the metro?

This specific example was made in Rome, Italy. There were two escalators but most people were eager to use the closer one. Furthermore, they use the second escalator in recognizable waves whenever the number of people waiting for the first one reaches a certain threshold.

There are lots of scales on which to model crowds.

- Microscopic – modeling every single pedestrian;
- Macroscopic – modeling the density of pedestrians on certain location;
- Mixed – combination of both approaches.

One has to be careful about continuum models, they can easily not make sense. We have to remember that there are some minimum size objects. When we are devising a continuous model, we are using approximations and thus we can introduce errors.

Different methods of solving differential equations give different errors:

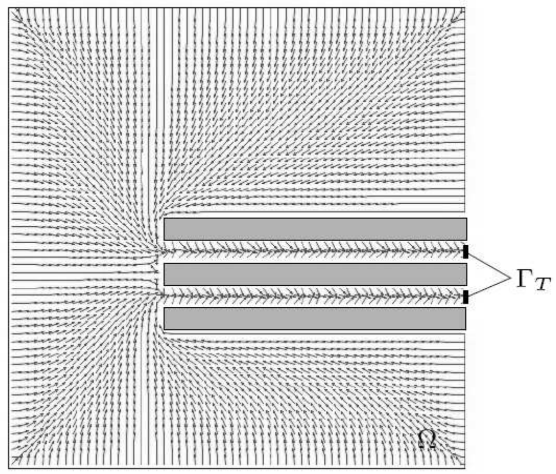

Fig. 3. Desired velocity field

- Euler estimate - first order error
- Runge-Kutta - fourth order error

On another example of model people were walking in different directions through a narrow door. Using the microscale model pedestrians get stuck and do not get through the door. On the other hand, in a macroscale model, pedestrians phantom through each other. Possible solution is to mix the scales (integrated scales). The simulation of a model using this integration looks plausible.

Interactions for self-organization: (organizations inside crowds)

3 New Mathematical Framework

We have to distinguish two continuum models:

- ODE — quantity depends only on time
- PDE — quantity depends on time and space

How to deal with it? We want to mix both macroscopic and microscopic scales. We then get the *mixed scale.* And that is the best.

First of all we need to define measures mathematically.

We use a *time-evolving measures model.*

A measure

$$\mu : (t, E) \to \mu(t, E)$$

represents the number of pedestrians in the region E.

Then define a flow map

$$\gamma : x \to x + v(x, \mu)\Delta t$$

that moves each point with given velocity.

At next time step measure is given by

$$\mu(t + \Delta t, E) = \mu(t, \gamma^- 1(E))$$

We require only local Lipschitz continuity for v_d

To measure errors, we need a metric on measures. One such metric with nice properties is the *Wasserstein metric*. Our problem has simple definition for the Dirac masses So basically we use a measure on product of a space instead of a map.

Let us denote the *pushforward measure* by $T\#\mu$. The *transportation cost* of mapping T is defined as

$$c_p(T) = \int_X d\big(x, T(x)\big)^p \mathrm{d}\mu_1(x).$$

Finally, the *p-th Wasserstein metric* is defined by taking the infimum over all mappings that transforms μ_1 into μ_2:

$$W_p(\mu_1, \mu_2) = \big(\inf\{c_p(T) \,\big|\, T\colon X \to X, T\#\mu_1 = \mu_2\}\big)^{1/p}.$$

Alternatively, the Wasserstein metric can be defined as the infimum of integral on product space

$$W_p(\mu_1, \mu_2) = \left(\inf_{\pi \in \Gamma(\mu_1, \mu_2)} \int_{X \times X} d(x,y)^p \mathrm{d}\pi(x,y) \right)^{1/p},$$

where $\Gamma(\mu_1, \mu_2)$ is the space of all measures on product space with marginals μ_1 and μ_2.

Theorem 1 (Transport Equation).

$$\partial_t \mu + \nabla [v[\mu] \cdot \mu] = 0$$

The mapping v maps every probabilistic measure to a vector field that is locally Lipschitz and locally bounded.

New result: Generalization for the case with a source of pedestrians.

The classical Wasserstein measure is well-defined only if the measures μ_1 and μ_2 have the same total measure. Moreover, it need not be symmetric in case of measures that are not absolutely continuous. Therefore, we introduce the *generalized Wasserstein metric*:

$$W_p^{a,b}(\mu_1, \mu_2) := \inf_{\substack{\tilde{\mu_1} \le \mu_1 \\ \tilde{\mu_2} \le \mu_2 \\ |\tilde{\mu_1}| = |\tilde{\mu_2}|}} a(|\mu_1 - \tilde{\mu_1}|^p + |\mu_2 - \tilde{\mu_2}|^p) + bW_p(\tilde{\mu_1}, \tilde{\mu_2})$$

Theorem 2 (Transport Equation with source).

$$\partial_t \mu + \nabla [v[\mu] \cdot \mu] = h[\mu]$$

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