

OPTIMIZATION METHODS: CLASS 3

Linearity, convexity, affinity

The exercises are on the opposite side.

D: A set $A \subseteq \mathbb{R}^d$ is an *affine space*, if A is of the form $L + v$ for some linear space L and a shift vector $v \in \mathbb{R}^d$. By “ A is of the form $L + v$ ” we mean a bijection between vectors of L and vectors of A given as $b(u) = u + v$. Each affine space has a *dimension*, defined as the dimension of its associated linear space L .

D: A vector x is an *affine combination* of a finite set of vectors $a_1, a_2, \dots, a_n$ if $x = \sum_{i=1}^n \alpha_i a_i$, where α_i are real number satisfying $\sum_{i=1}^n \alpha_i = 1$.

A set of vectors $V \subseteq \mathbb{R}^d$ is *affinely independent* if it holds that no vector $v \in V$ is an affine combination of the rest.

D: Given a set of vectors $V \subseteq \mathbb{R}^d$, we can think of its *affine span*, which is a set of vectors A that are all possible affine combinations of any finite subset of V .

Similar to the linear spaces, affine spaces have a finite basis, so we do not need to consider all finite subsets of V , but we can generate the affine span as affine combinations of the base.

D: A *hyperplane* is any affine space in $\mathbb{R}^d$ of dimension $d - 1$. Thus, on a 2D plane, any line is a hyperplane. In the 3D space, any plane is a hyperplane, and so on.

A hyperplane splits the space $\mathbb{R}^d$ into two *halfspaces*. We count the hyperplane itself as a part of both halfspaces.

D: A set $K \subseteq \mathbb{R}^d$ is a *convex set*, if $\forall x, y \in K, \forall t \in [0, 1] : tx + (1 - t)y \in K$. In other words, if you take two points inside the convex set K , the entire line segment between those two points must belong to K .

D: A vector x is a *convex combination* of a set of vectors $a_1, a_2, \dots, a_n$ if $x = \sum_{i=1}^n \alpha_i a_i$, where α_i are real numbers satisfying $\sum_{i=1}^n \alpha_i = 1$ and also $\forall i : \alpha_i \in [0, 1]$.

A set of vectors/points $V \subseteq \mathbb{R}^d$ is *in a convex position*, if it holds that no vector $v \in V$ is a convex combination of the rest.

D: As with linearity and affinity, for convexity we also define a span/hull:

If we have a set of vectors $V \subseteq \mathbb{R}^d$, its *convex hull* is a set of all vectors C , which are convex combinations of any finite subset of the vectors in V .

Here, we really need to consider any finite subset of V , because convex sets in general do not have a finite basis.

D: A *convex polytope* is any object in $\mathbb{R}^d$ that is an intersection of finitely many halfspaces. Alternatively, we can say that a convex polytope is any set of points of the form $\{x | Ax \leq b\}$ for some real matrix A and some real vector b .

A quick reminder from linear algebra:

T: Every linear space of dimension k contains a basis of k vectors. We can find a special basis that is *orthogonal* or even *orthonormal*. And for any basis (even a non-orthogonal one) we can compute its *orthogonal complement*. (How?)

From last time:

EXERCISE ONE Josef K. got an exercise at his Optimization methods class:

Design an integer program for the travelling salesman problem: For a given graph with distances $G = (V, E, f)$, where $f : E \rightarrow \mathbb{R}_0^+$, find a Hamiltonian cycle with the shortest length.

He suggests the following:

“For every edge uv we have a variable $x_{uv} \in \{0, 1\}$, the target function is $\min \sum_{uv \in E} f(uv)x_{uv}$ and for every vertex u we create a condition of the form $\sum_{i|ui \in E} x_{ui} = 2$.”

Prove that Josef K. got the right solution – or prove him wrong and suggest a better one.

EXERCISE TWO Let us consider a polytope (actually, a line segment)

$$P = \{x \in \mathbb{R} | x \geq 1 \& x \leq 2\}.$$

Transform its inequalities into an equational form and then draw the polytope in the equational form.

EXERCISE THREE Prove the following equivalence:

A set of $n + 1$ vectors $v_0, v_1, v_2, v_3, \dots, v_n$ in $\mathbb{R}^d$ is affinely independent if and only if the set of n vectors $v_1 - v_0, v_2 - v_0, v_3 - v_0, \dots, v_n - v_0$ is linearly independent.

EXERCISE FOUR Alice and Bob play a game. Alice will think of a linear inequality in $\mathbb{R}^3$ but it will not describe it to Bob. She will only tell Bob three points b_1, b_2, b_3 in $\mathbb{R}^3$, which satisfy the inequality.

Bob now must call out new points $b_4, b_5, b_6 \dots$ which also satisfy the inequality – until Alice gets bored of the game and they both go play hopscotch.

Suggest a strategy for Bob to win.

EXERCISE FIVE

1. Can two 2D planes intersect in exactly one point, if we place them in $\mathbb{R}^4$?
2. *If that was too easy:* Can two 3D spaces (affine spaces of dimension 3) intersect in exactly one point in $\mathbb{R}^5$?

EXERCISE SIX

1. Prove that each affine space can be expressed as an intersection of finitely many affine hyperplanes.
2. Prove that every hyperplane can be expressed as the set $\{x | c^T x = b\}$.

Hint: Whenever you want to prove something about an affine space, try shifting it using the vector $-v$ so that it becomes a linear space L , and then argue about the linear space instead.

EXERCISE SEVEN Prove the following equivalence, which gives you an easy way to algebraically describe affine spaces:

A set $F \subseteq \mathbb{R}^d$ is an affine space if and only if $F = \{x \in \mathbb{R}^d | Ax = b\}$ for some matrix $A \in \mathbb{R}^{d \times d}$ and some vector $b \in \mathbb{R}^d$. (Just to make things simpler, we count an empty set as an affine space too.)

EXERCISE EIGHT We know that a set K is convex if the set contains all line segments with endpoints in K . Prove a very similar description for affinity:

A set A is an affine subspace of $\mathbb{R}^d$ if and only if for each two points $a, b \in A$ the entire *line* defined by a, b is contained in A .