Limiting Free Boundary Problems
The One-Dimensional Profile

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- $\partial\Omega$ is a smooth compact hypersurface in $\mathbb{R}^n$
- L is a linear elliptic operator
- $f : \partial\Omega \to \mathbb{R}$ is a nonnegative function and $g : \bar{\Omega} \to \mathbb{R}$ is a positive function

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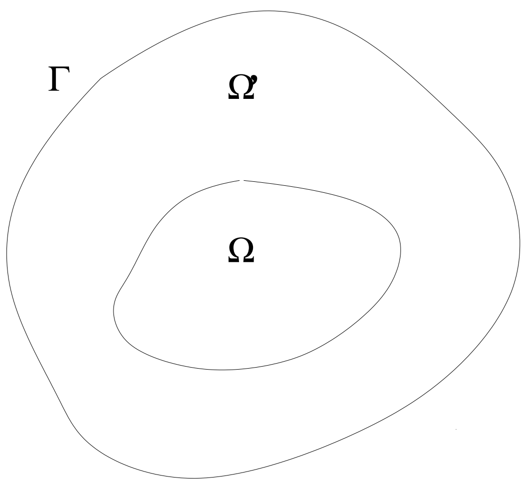
Limiting Free Boundary Problem: Picture



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Let β be a function positive in $(0,1)$ with support in $[0,1]$. Furthermore, assume that

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This is an approximation to the Dirac Delta function.

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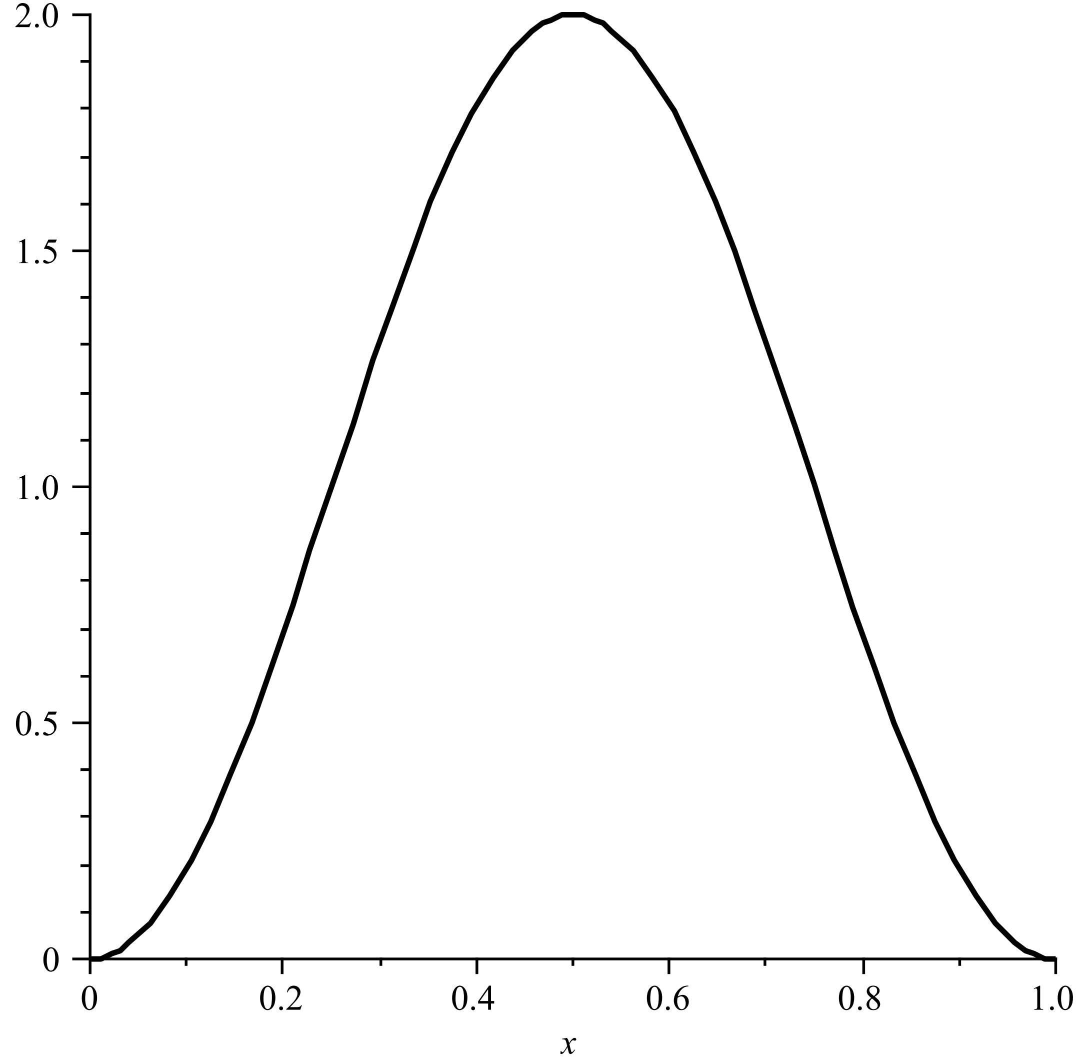

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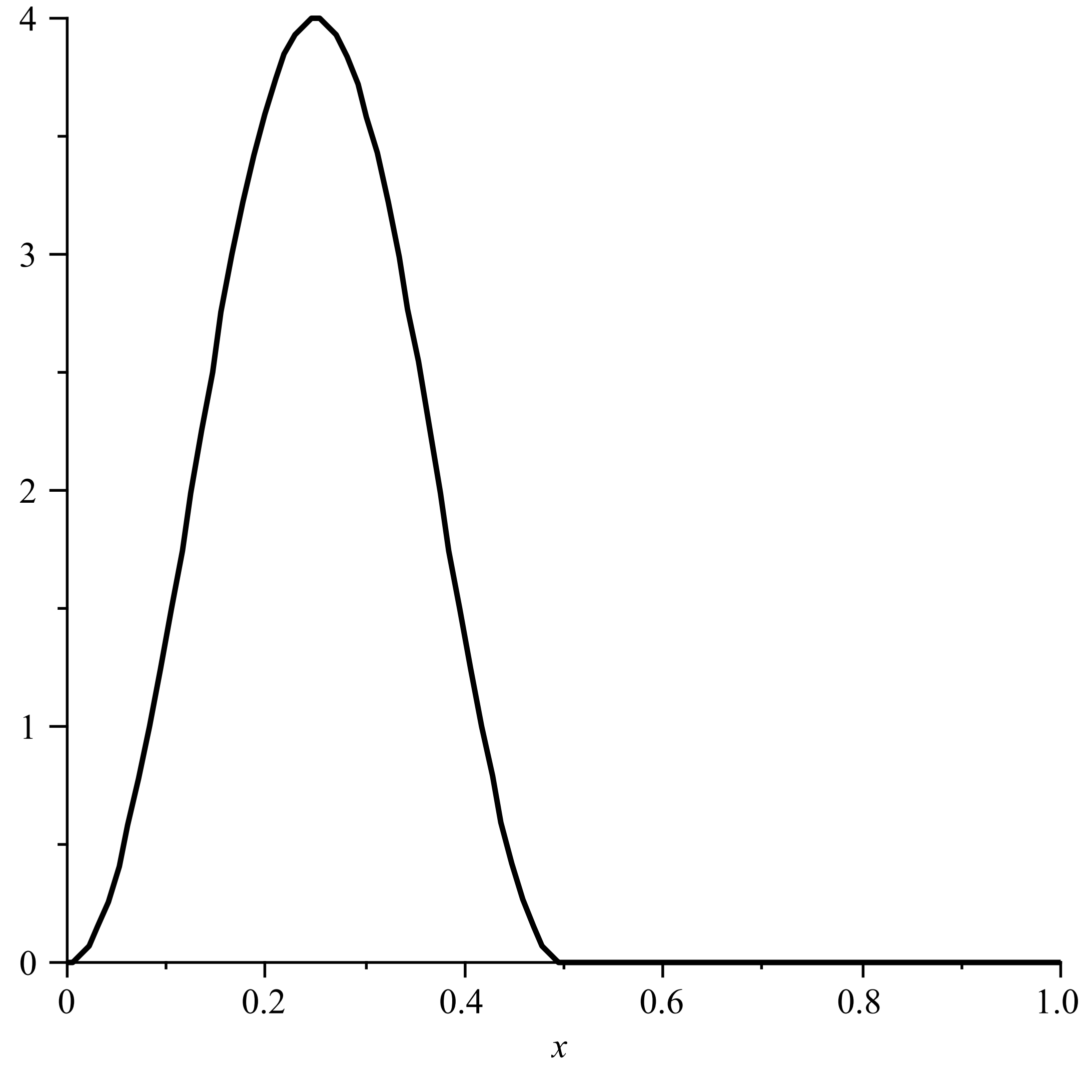

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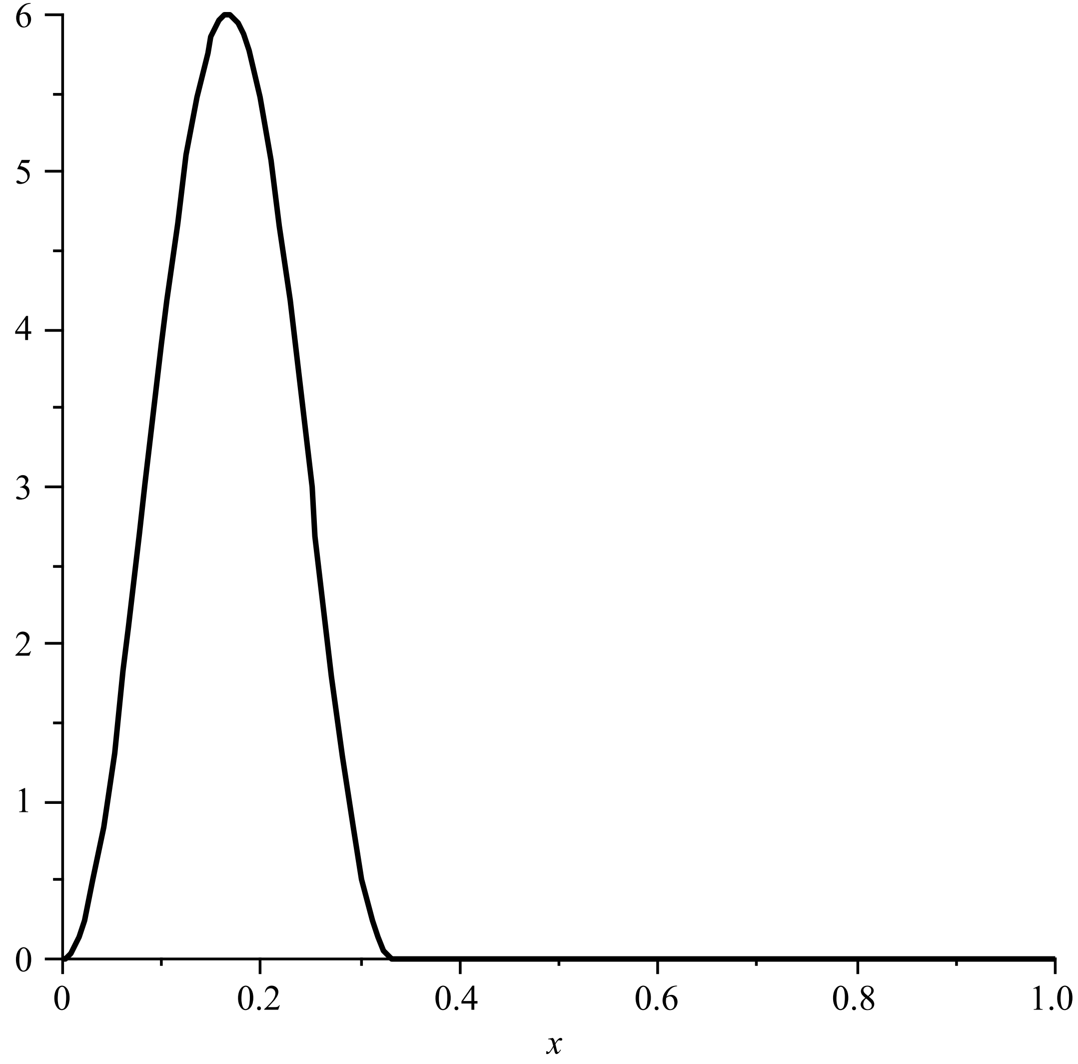

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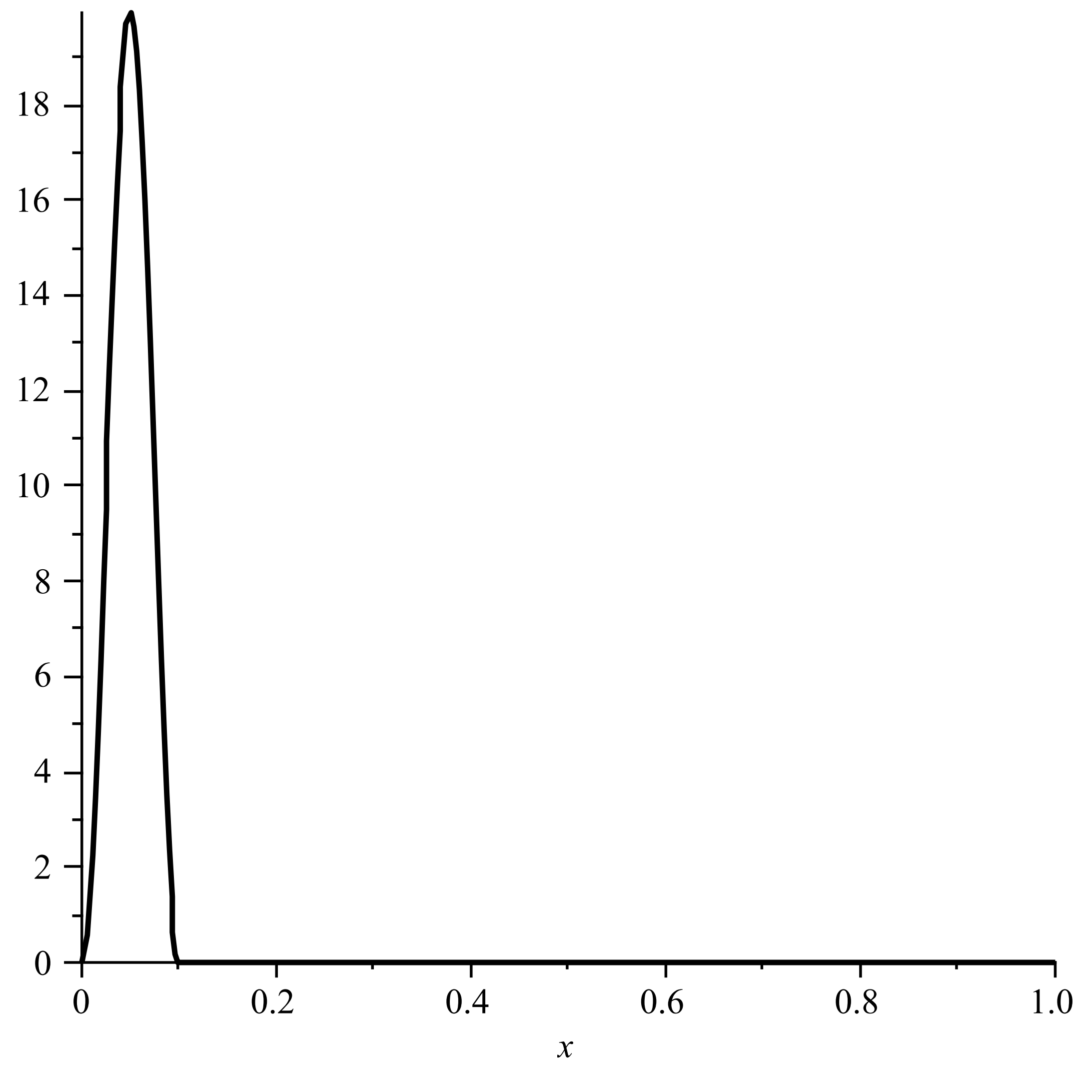

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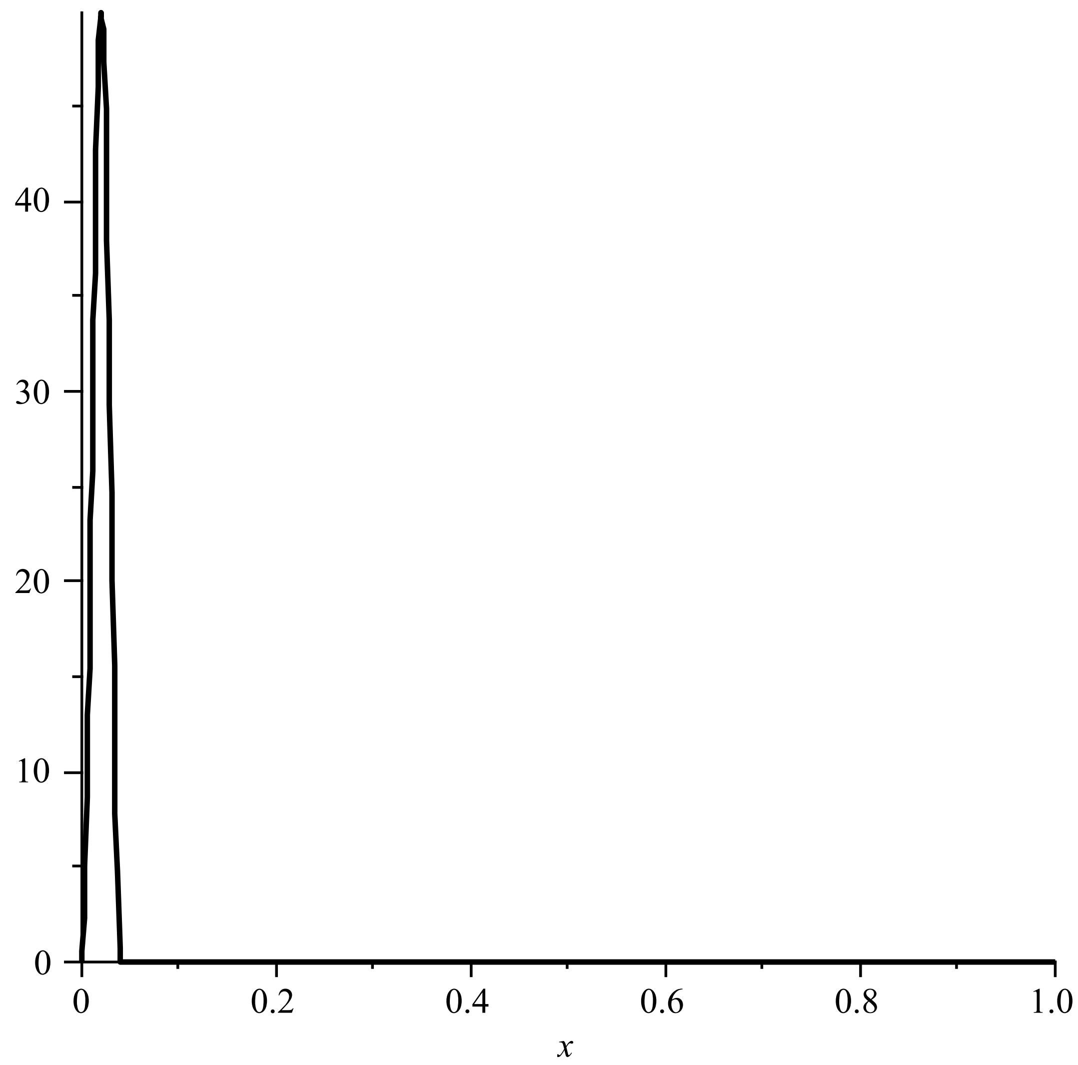

Finding the Boundary

One way to find possible solutions to the overdetermined elliptic boundary value problem is to consider the limit of the approximating regularized problem

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If

$$u_0 := \lim_{\epsilon \to 0} u_\epsilon,$$

we take Γ to be $\partial\{u_0 > 0\}$.

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- D. Moreira and Eduardo Teixeira obtained a thorough description of the limiting free boundary problem for linear equations in divergence form very recently (2007)

Ongoing Research

Current research in this field considers equations of the form

$$\begin{cases} F(D^2 u_\epsilon, x) = g^2(x)\beta_\epsilon(u_\epsilon) & \text{in } \Omega \\ u_\epsilon = f & \text{on } \partial\Omega, \end{cases}$$

where F is a fully nonlinear elliptic operator.

The One-Dimensional Profile

It is useful to restrict the analysis of the limiting free boundary and consider the one-dimensional profile by analyzing the limiting configuration for

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Ellipticity in one dimension means that there exist constants $0 < \lambda \leq \Lambda < +\infty$ such that

$$\lambda b \leq f(a+b) - f(a) \leq \Lambda b$$

for all nonnegative reals b.

The One-Dimensional Profile

By defining $v_\epsilon(x) := \frac{1}{\epsilon} u(\epsilon x)$, we obtain

$$\epsilon f(\frac{1}{\epsilon} v_\epsilon'') = \beta(v_\epsilon).$$

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We now turn our analysis to f_ϵ.

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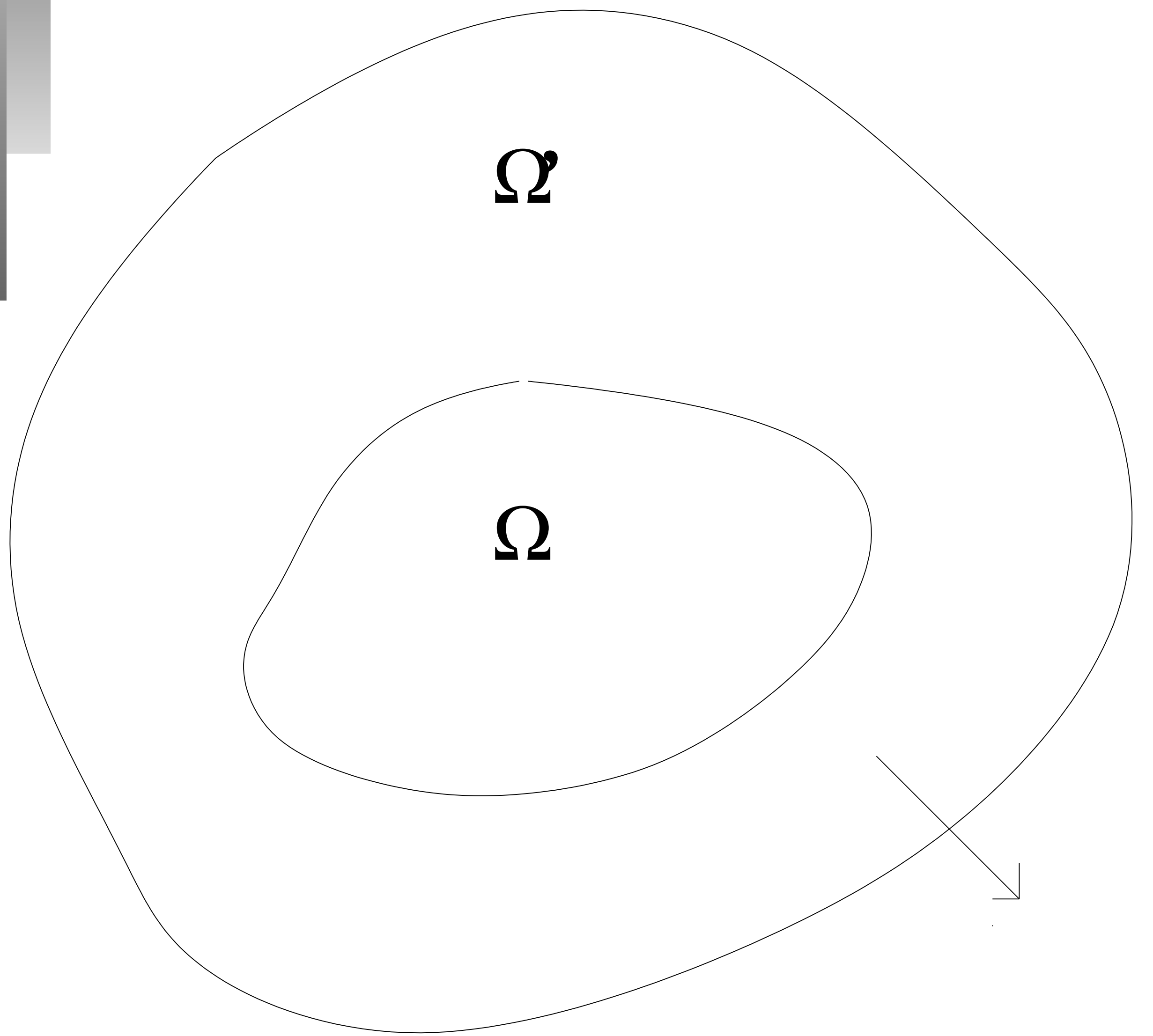

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and assume that there exist constants $0 < \lambda \leq \Lambda < \infty$ such that

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for all $a, b \geq 0$.

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Note that f_ϵ is uniformly elliptic with the same ellipticity constants as f.

The Problem

Our initial conjecture was that there exists a function $f^\star$ satisfying the ellipticity condition such that

$$f^\star(s) := \lim_{\epsilon \longrightarrow 0} f_\epsilon(s)$$

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converges uniformly over compact sets.
Note: because of the Arzelá-Ascoli Theorem, we only need to consider pointwise convergence.

The Problem

However, this conjecture does not hold for all f. We will see examples of this later on.

Uniform Ellipticity

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- It is strictly increasing
- All secants and derivatives are bounded by the ellipticity constants

Examples

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Does $f^\star$ exist? Yes.

$$\lim_{\epsilon \to 0} \epsilon c(\frac{s}{\epsilon}) = cs,$$

so $f = f^\star$.

Convex Functions

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Suppose f is convex. Then, as $\epsilon \to 0$, $f_\epsilon(s)$ is strictly increasing. It is also bounded by Λ, so we know it converges.

Therefore, if f is convex, then $f^\star$ exists.

L'Hospital's Rule

Consider functions f where

$$\lim_{n \to \infty} f'(n)$$

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Then, by L'Hospital's Rule, $f^{\star}(s) = \xi s$.

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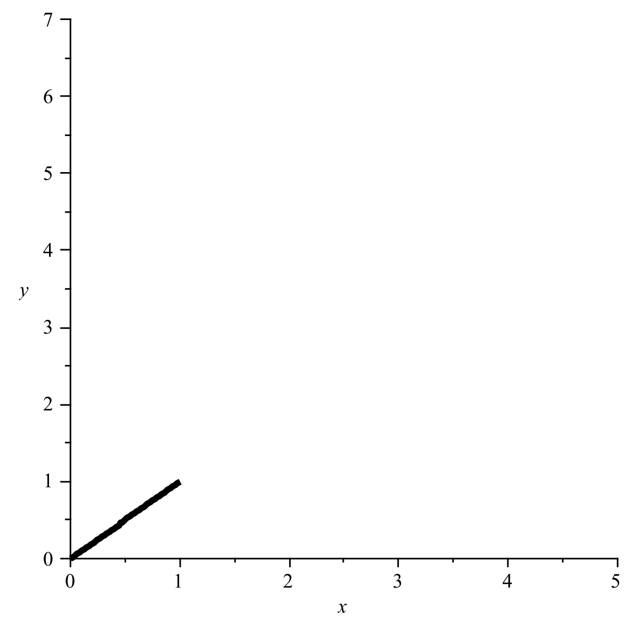

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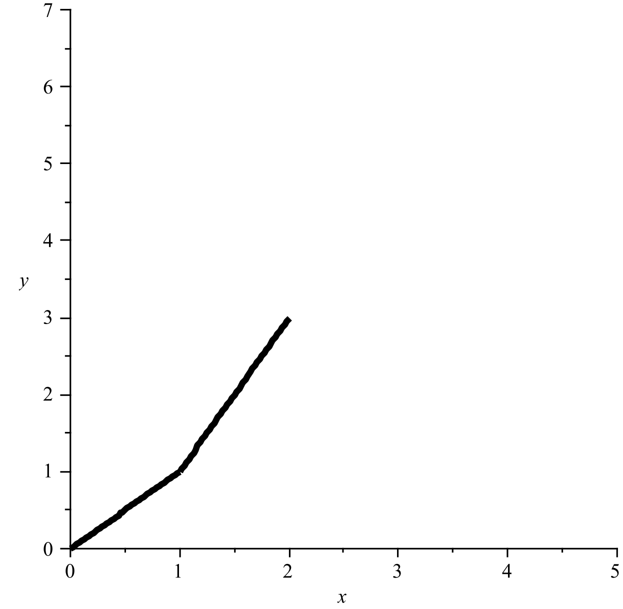

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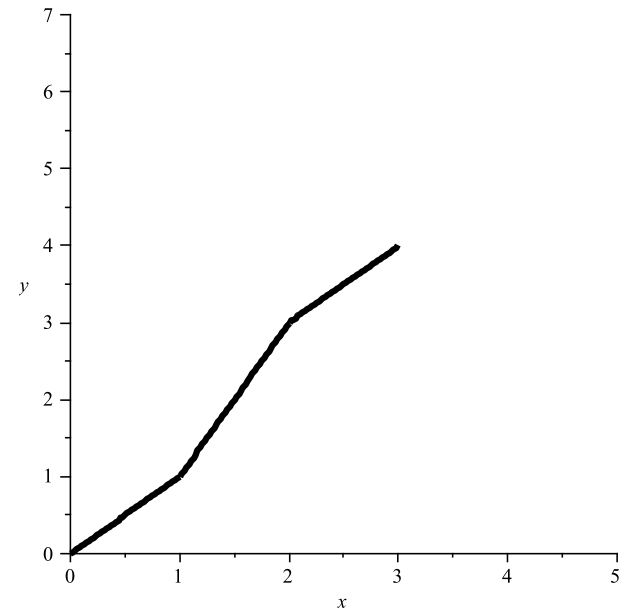

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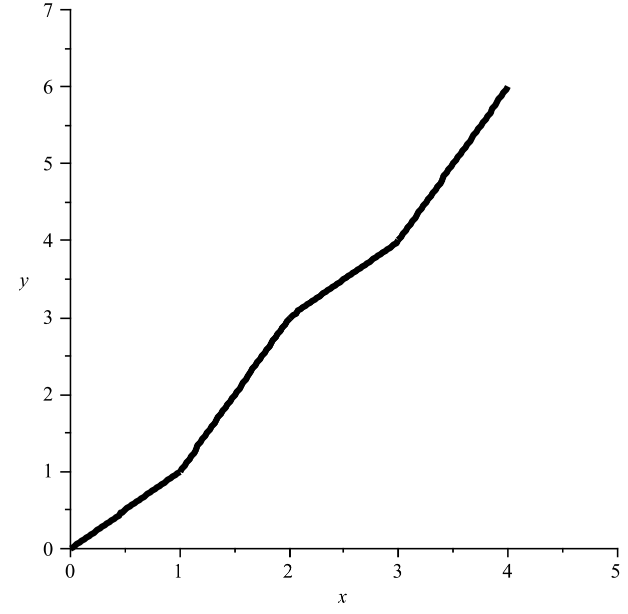

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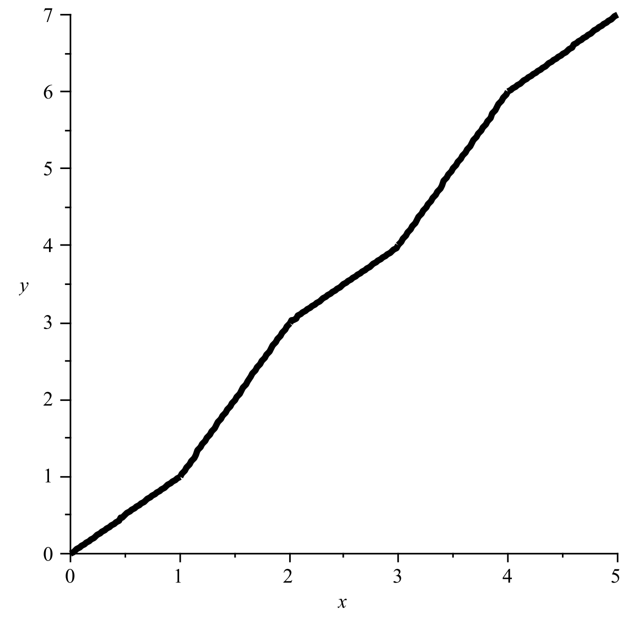

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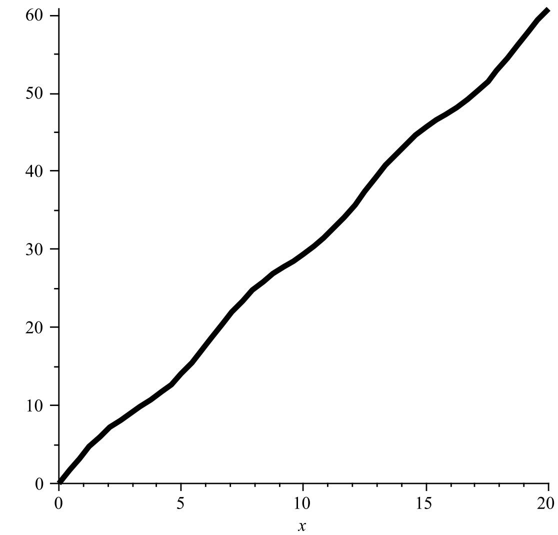

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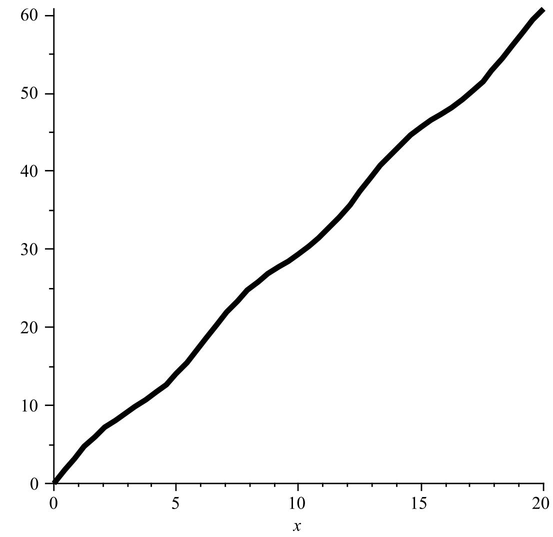

However, this does not mean $f^\star$ does not exist. In fact,

$$f^\star(s) = 3s.$$

A (True) Theorem

Define

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Theorem: The function $f^\star(s)$ exists if and only if α exists. Furthermore, if α exists, then $f^\star(s) = \alpha s$.

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This theorem says that whenever $f^\star$ exists, it is linear.

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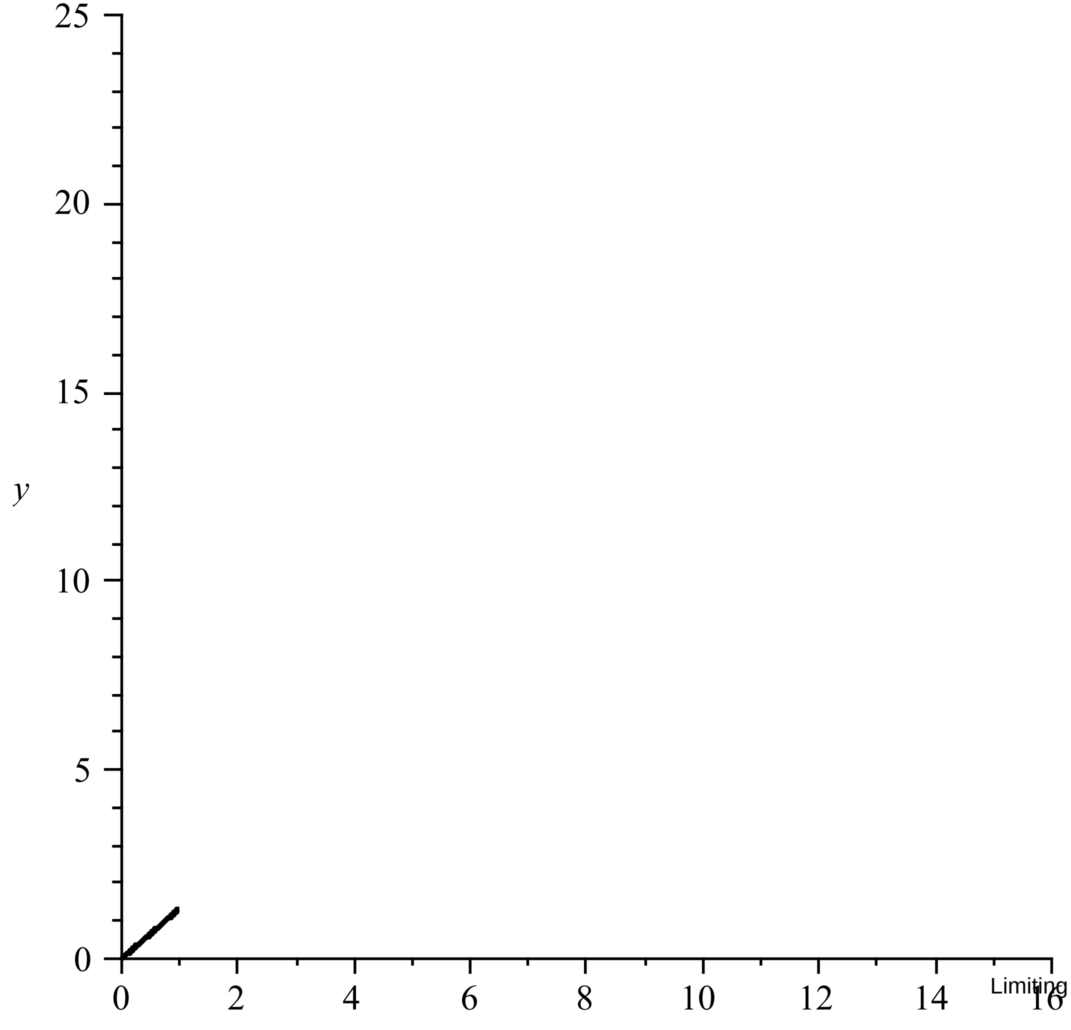

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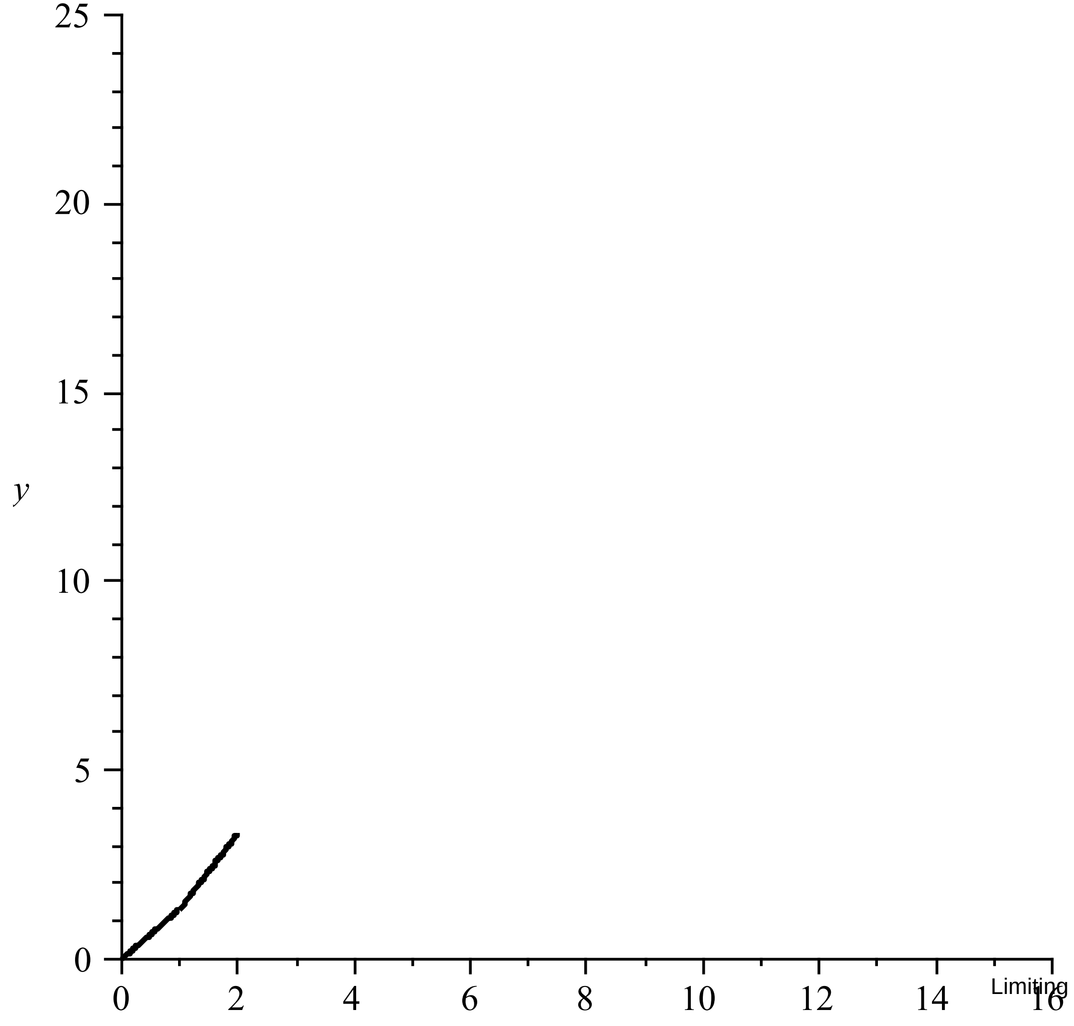

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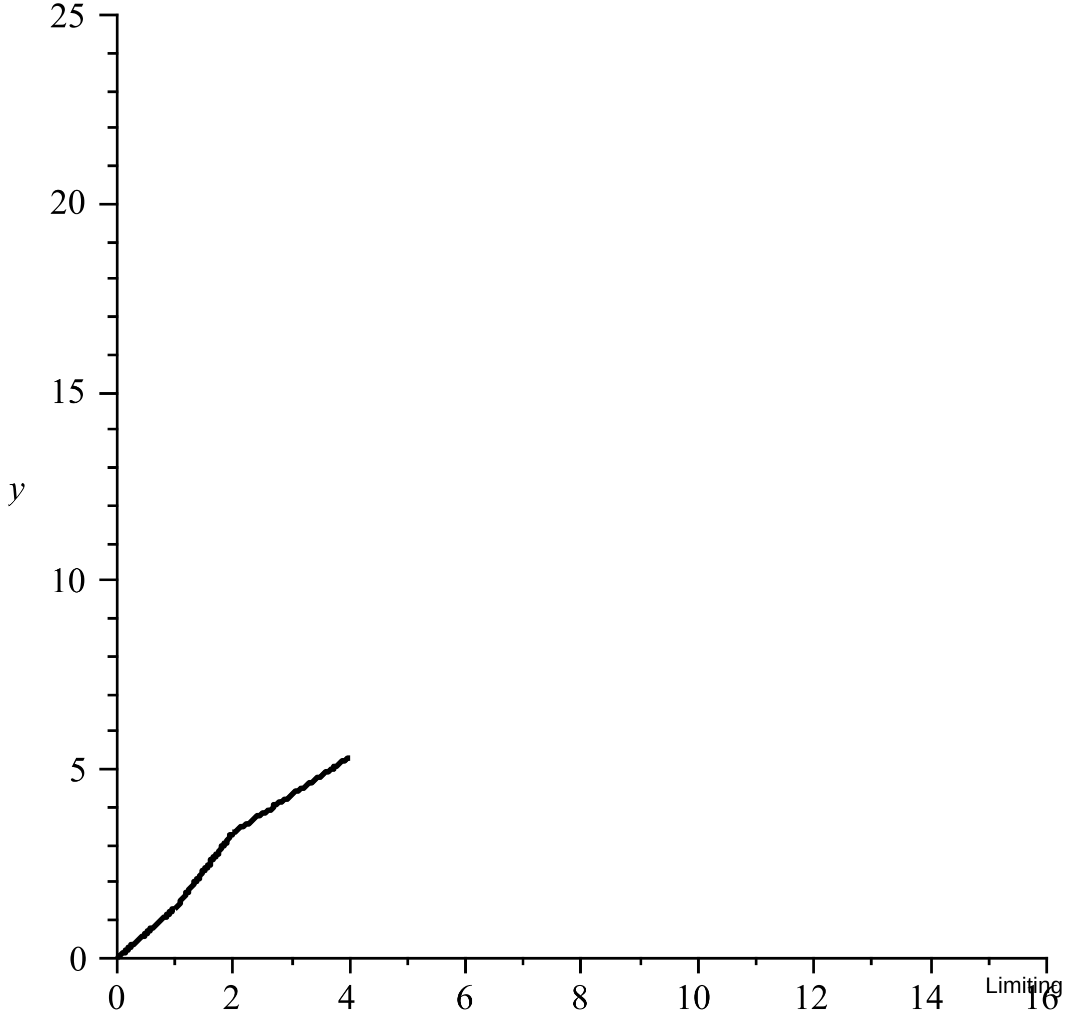

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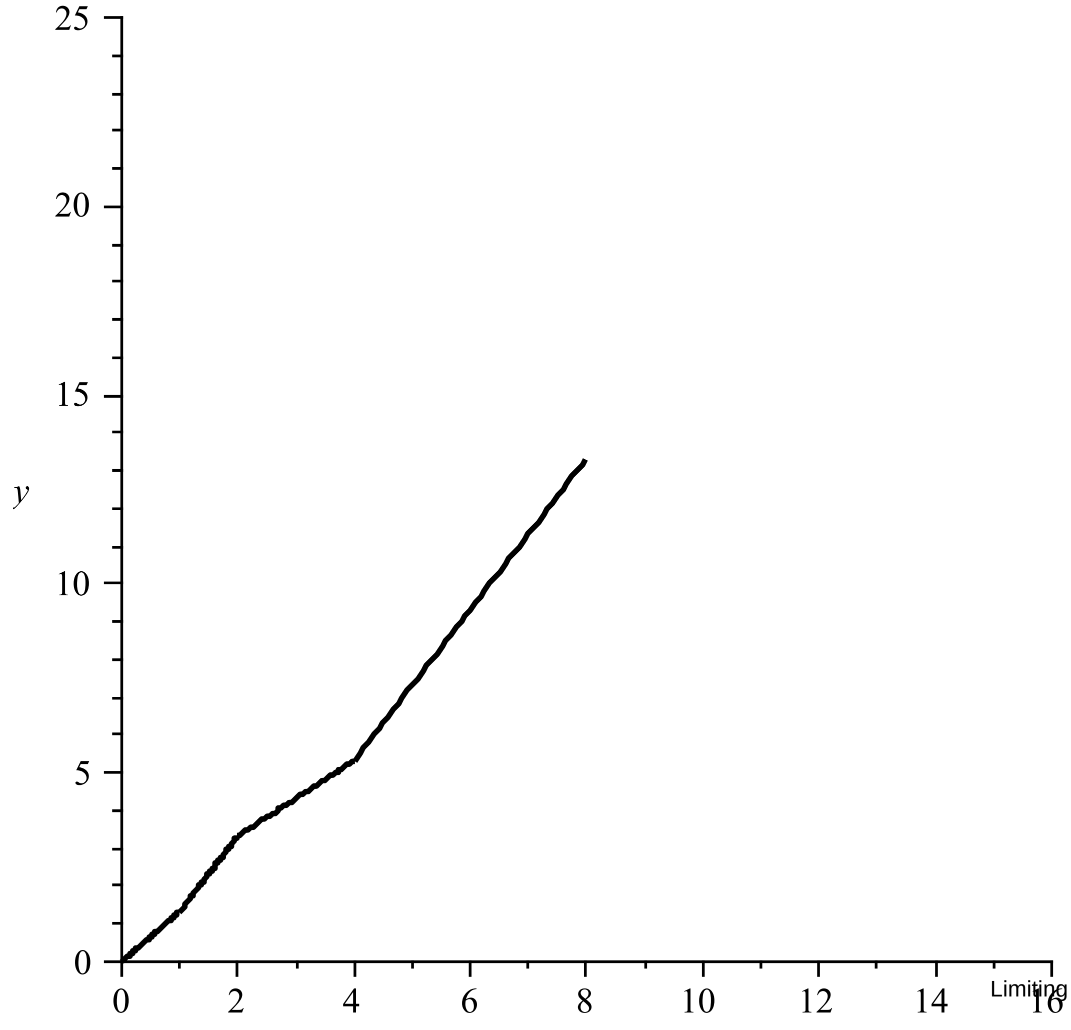

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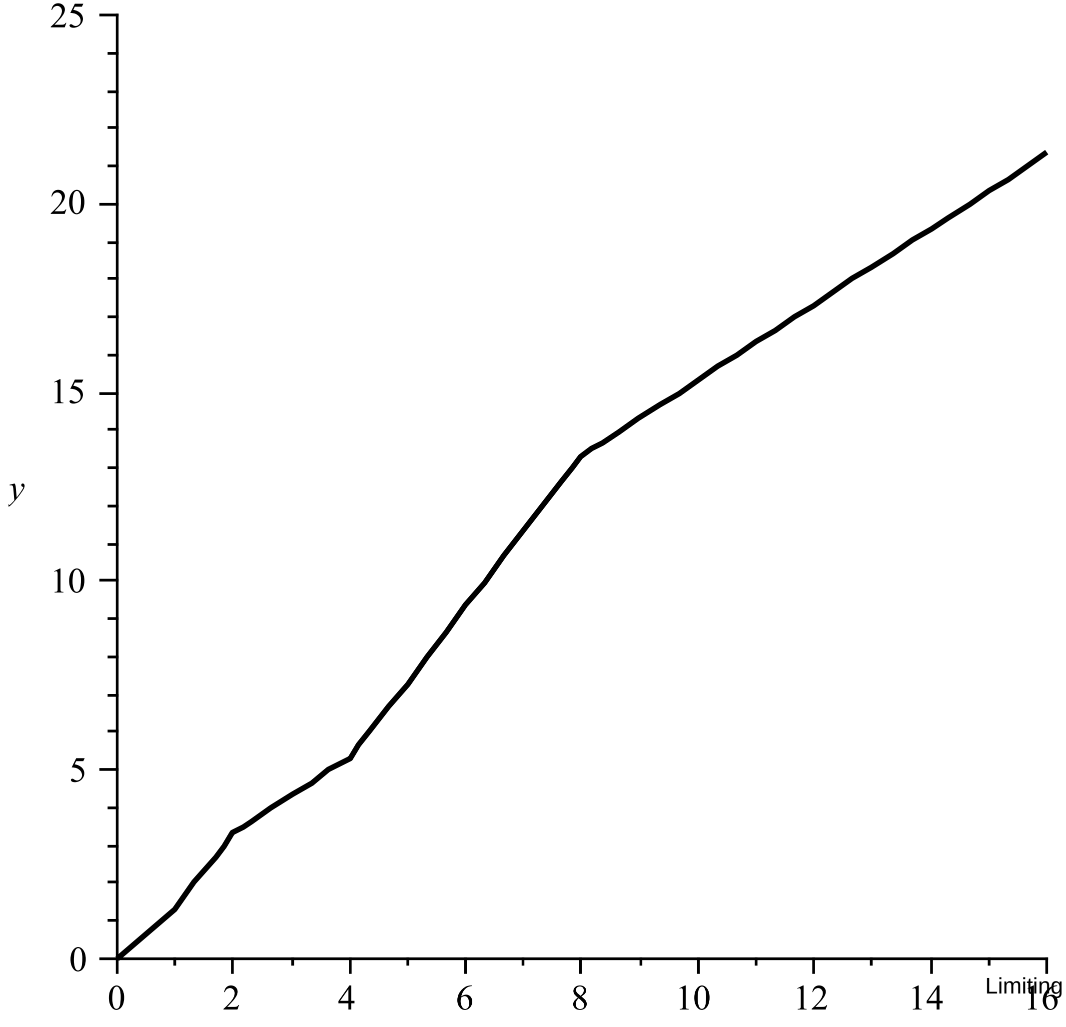

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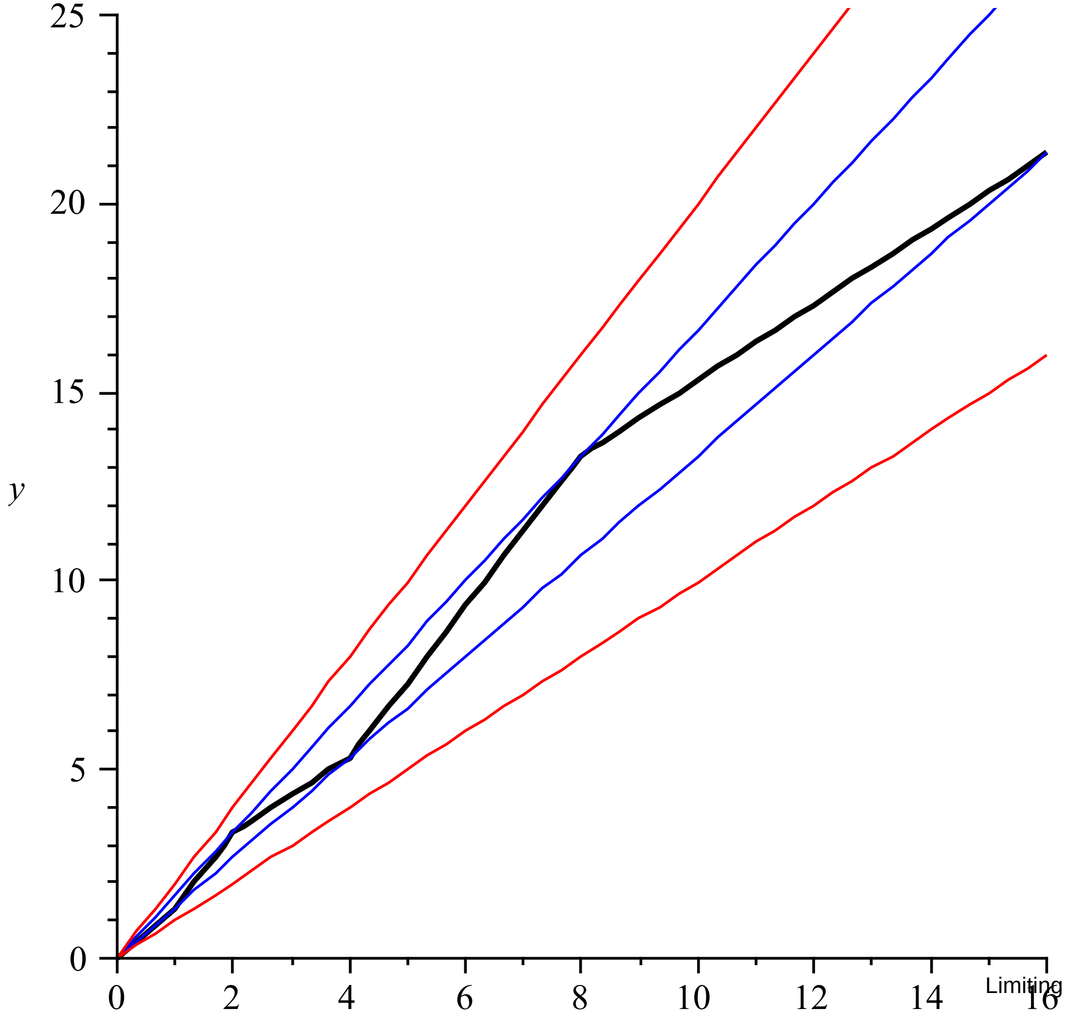

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As ϵ approaches 0, f_ϵ oscillates between $\frac{4}{3}$ and $\frac{5}{3}$, so the limit does not exist.

Future Research

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Let $F : \mathcal{S}(n) \times \Omega \to \mathbb{R}$ be a fully nonlinear elliptic operator, where $\mathcal{S}(n)$ is the space of n by n symmetric matrices.

Future Research

There is also a higher dimensional version of this problem.
Let $F : \mathcal{S}(n) \times \Omega \to \mathbb{R}$ be a fully nonlinear elliptic operator, where $\mathcal{S}(n)$ is the space of n by n symmetric matrices.
Furthermore, suppose the following ellipticitiy condition holds: there exist constants $0 < \lambda \leq \Lambda < +\infty$ such that

$$F(M+N,x) \leq F(M,x) + \Lambda \parallel N^+ \parallel - \lambda \parallel N^- \parallel$$

$$\forall M, N \in \mathcal{S}(n), \forall x \in \Omega.$$

Future Research

What happens in this more general case?