Connectedness and diameter of dual - summary

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For given oriented graphs G and H, $G \to H$ will denote the existence of a homomorphism from G to H and $G \not\to H$ its non-existence. The *dual* of T is a graph D_T such that

$$T \to G \Longleftrightarrow G \not\to D_T \tag{1}$$

holds for every oriented graph G. Nešetřil and Tardif have proved that D_T exists if and only if T is a tree (and it is unique up to homomorphism equivalence). They have also found the following construction of D_T.

Proposition 1 *D_T defined by*
$V(D_T) = \{f : V(T) \to V(T); (u, f(u)) \in E(T)$ *or* $(f(u), u) \in E(T)$ *for all* $u \in V(T)\}$, $E(D_T) = \{(f, g);$ *for all* $(u, v) \in E(T)$ *we have* $f(u) \neq v$ *or* $g(v) \neq u\}$
is a dual of T.

Besides having the property (1) and thus being useful in many situations, dual defined in proposition 1 is also an interesting combinatorial structure, which is certainly worth studying. Duals can be exponentially large (in the number of vertices of the original trees) and constructing examples is complicated even for small trees; that's why almost nothing is known about their structure and properties.

In my bachelor thesis I investigated properties of the dual constructed in Proposition 1. I found the following:

Theorem 1 *Let T be an oriented tree. After removing isolated vertices, its dual D_T (as defined in Proposition 1) is connected. Moreover, its diameter is at most $O(n^2)$.*

Both proofs are by building T from K_1 (by adding leaves) and investigating how D_T changes in each step.

Sketch of proof:

Definition 1 *Vertex $u \in V(T)$ is a* source if its indegree is zero. It is a problematic source for $f \in V(D_T)$ if for all its neighbors w we have $f(w) = u$. Similarly, u is a sink if its outdegree is zero and it is a problematic sink for $f \in V(D_T)$ if $f(w) = u$ for all vertices w adjacent to u.

Lemma 1 *(Characterization of isolated vertices of dual)*

Outdegree of f in D_T is zero if and only if there exists a problematic sink for f in T. Indegree of f in D_T is zero if and only if there exists a problematic source for f in T.

If we create T' from T by adding a leaf, then $D_{T'}$ contains D_T as a subgraph. Besides it contains many "new" vertices (not included in a copy of D_T) and edges between "old" and "new" vertices. However, there are no new edges joining two "old" vertices and no edges joining two "new" vertices. More precise formulation of this observations follows in next lemmas. First, let's consider the case when the newly added edge (joining T with the new leaf) is directed outwards from T.

Definition 2 *Let T be an oriented tree, D_T its dual constructed in Proposition 1, $V(T) = \{\tilde{f}_1, \ldots, \tilde{f}_k\}$, and $u \in V(T)$. We will define T' by putting $V(T') = V(T) \cup \{v\}$ and $E(T') = E(T) \cup \{(u,v)\}$. We will define mappings $f_1, \ldots, f_k : V(T') \to V(T')$ such that $f_i \restriction V(T) = \tilde{f}_i$ and $f_i(v) = u$ for $i = 1, \ldots, k$. Similarly, $f'_1, \ldots, f'_k$ will be mappings $V(T') \to V(T')$ such that $f'_i(w) = \tilde{f}_i(w)$ for $w \in V(T) \setminus \{u\}$, $f'_i(u) = v$ and $f'_i(v) = u$, $i = 1, \ldots, k$.*

Lemma 2 *Mappings $\{f_1, \ldots, f_k, f'_1, \ldots, f'_k\}$ defined above are precisely the vertices of $D_{T'}$.*

Lemma 3 *Let T be an oriented tree, $V(D_T) = \{\tilde{f}_1, \ldots, \tilde{f}_k\}$. Let T', f_i and f'_i ($i = 1, \ldots, k$) be as in definition 2. Then*

- *outdegree of f'_i is 0 for every $i = 1, \ldots, k$*
- *$(\tilde{f}_i, \tilde{f}_j)$ is an edge of D_T if and only if (f_i, f_j) is an edge of $D_{T'}$*
- *if (f_i, f_j) is an edge of $D_{T'}$, then (f_i, f'_j) is an edge of $D_{T'}$.*

Theorem 2 *If D_T is connected after removing isolated vertices, then $D_{T'}$ has this property as well.*

Proof. If D_T is connected except for isolated vertices, it has at most one non-trivial connected component, say K. All f_i such that $\tilde{f}_i \in K$ belong to the same connected component of $D_{T'}$ (see lemma 3); we will call this component K'. Lemma 3 also says that all new edges of $D_{T'}$ (those not contained in the copy of D_T) are of the shape (f_i, f'_j) for some i and j; there are no new edges (f'_i, f'_j), (f'_i, f_j) or (f_i, f_j). If $\tilde{f}_i$ is not isolated in D_T, then $f_i \in K'$ and adding an edge (f_i, f'_j) doesn't jeopardize the connectedness. There can be a problem only if $\tilde{f}_i$ is isolated.

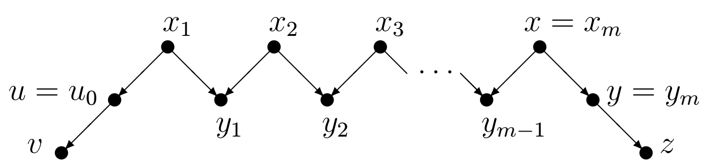
So we are interested in such vertices f_i that $\tilde{f}_i$ is isolated but f_i is not. T has at least one problematic sink and at least one problematic source for these $\tilde{f}_i$ (lemma 1). But since f_i is not isolated in $D_{T'}$, there is exactly one problematic sink for $\tilde{f}_i$, namely u. (If there were more problematic sinks for $\tilde{f}_i$ in T, they would be problematic for f_i as well.) This has also the consequence that if u is not sink, then f_i is isolated in D'_T if and only if $\tilde{f}_i$ is isolated in D_T, and dual stays connected after adding (u, v).

Let $\tilde{f}$ be such mapping that T has only one problematic sink for $\tilde{f}$, and this sink is u. After adding the edge (u, v), u is not sink any more, and thus (by lemma 1) there exists $g \in \{f'_1, \ldots, f'_k\}$ for which (f, g) is an edge of $D_{T'}$. We will define mapping $\tilde{f}^*$: it is equal to $\tilde{f}$ on the set of sources, and it maps the rest of the vertices of T (not sources) arbitrarily against the direction of some edge adjacent to them. Clearly, if (f, g) is an edge, then (f^*, g) is an edge as well.

If u is not problematic for $\tilde{f}^*$, then outdegree of $\tilde{f}^*$ is greater than zero, thus $\tilde{f}^* \in K$ and $f^* \in K'$. (f, g) and (f^*, g) are edges, so both f and g belong to K'. If u is problematic for $\tilde{f}^*$, then all neighbors of u must be sources. We will distinguish two cases:

(1) There exist vertices x, y, z; $y \neq u$ in T such that $(x, y) \in E(T)$ and $(y, z) \in E(T)$ (in other words, $P_2 \to T$). Symmetrization of the oriented tree T is an undirected tree, thus it contains exactly one path with endpoints u and y. Let l_y denote the length of this path. Similarly, let l_z be the length of the path in $\mathrm{sym}(T)$ with endpoints u and z. If there are more such triplets (x, y, z), we will choose the one for which $\min(l_y, l_z)$ is minimal. We know that $y \neq u$ and since all neighbors w_i of u are sources, we have $z \neq u$, $w_i \neq y$ and $w_i \neq z$, thus $\min(l_y, l_z) \geq 2$.

(i) Let $l_y < l_z$. Then the path from u to y is a sequence of vertices $(y_0 = u, x_1, y_1, \ldots, x_{m-1}, y_{m-1}, x_m = x, y_m = y)$ with $(x_i, y_{i-1}) \in E(T)$ and $(x_i, y_i) \in E(T)$ for $i = 1, \ldots, m$ (where $m = l_y/2$). Here is what it looks like:



Let $k \leq m$ be the greatest integer such that $f^*(x_i) = y_{i-1}$ for $i = 1, \ldots, k$. Since u is a problematic sink for $\tilde{f}^*$, we have $k \geq 1$. Remember that we suppose $(f^*, g) \in E(D_{T'})$ for some g. To simplify notation in the next few paragraphs, we need to define two sets for each $f \in V(D_{T'})$, A_f and B_f, at this point: $A_f = \{(u, v) \in E(T'); f : u \mapsto v\}$ and $B_f = \{(u, v) \in E(T'); f : v \mapsto u\}$. Clearly $(f, g) \in E(D'_T)$ for some f and g if and only if $A_f \cap B_g = \emptyset$. If $k \leq m$, we will define g_1: $g_1(y_k) = x_{k+1}$, $g_1(w) = g(w)$ for $w \in V(T') \setminus \{y_k\}$. We can see that $B_{g_1} \subseteq B_g \cup \{(x_{k+1}, y_k)\}$. Since (f^*, g) is an edge, we have $A_{f^*} \cap B_g = \emptyset$. By definition of k we obtain $f^*(x_{k+1}) \neq y_k$, so $(x_{k+1}, y_k) \notin A_{f^*}$, and thus (f^*, g_1) is an edge. If $k = m$, we will define g_1 in a slightly different way: $g_1(y_k) = z$,

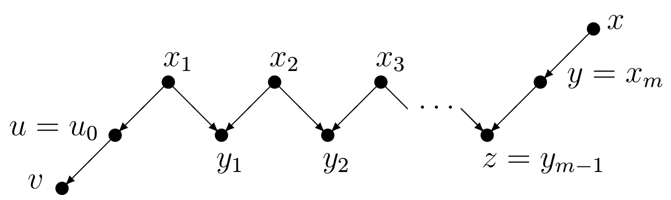
$g_1(w) = g(w)$ for $w \in V(T') \smallsetminus \{y_k\}$. Here $B_{g_1} \subseteq B_g$, and thus the edge (f^*, g_1) exists.

Next we will define f_1: $f_1(x_k) = y_k$, $f_1(w) = f^*(w)$ for $w \in V(T') \smallsetminus \{x_k\}$. $A_{f_1} \subseteq A_{f^*} \cup \{(x_k, y_k)\}$ and since $(x_k, y_k) \notin B_{g_1}$ and $A_{f^*} \cap B_{g_1} = \emptyset$, (f_1, g_1) is an edge of $D_{T'}$.

For $i = 2, \ldots, k$ we construct g_i and f_i stepwise so that $g_i(y_{k-i+1}) = x_{k-i+2}$, $g_i(w) = g_{i-1}(w)$ for $w \in V(T') \smallsetminus \{y_{k-i+1}\}$ and $f_i(x_{k-i+1}) = y_{k-i+1}$, $f_i(w) = f_{i-1}(w)$ for $w \in V(T') \smallsetminus \{x_{k-i+1}\}$. Again we have $B_{g_i} \subseteq B_{g_{i-1}} \cup \{(x_{k-i+2}, y_{k-i+1}\}$ and since $(x_{k-i+2}, y_{k-i+1}) \notin A_{f_{i-1}}$ and (f_{i-1}, g_{i-1}) is an edge, (f_{i-1}, g_i) is an edge as well. Similarly, $A_{f_i} \subseteq A_{f_{i-1}} \cup \{(x_{k-i+1}, y_{k-i+1}\}$ and since $(x_{k-i+1}, y_{k-i+1}) \notin B_{g_i}$ and $(f_{i-1}, g_i) \in E(D_{T'})$, also $(f_i, g_i) \in E(D_{T'})$.

We have $f_k(x_j) = y_j$ for $j = 1, \ldots, k$, $f_k(x_{k+1}) \neq y_k$ (if $k < m$) and f_k is equal to f^* on the rest of the vertices, so there are no problematic sinks for $\widetilde{f_k}$ in T', thus $f_k \in K'$. $(f, g, f^*, g_1, f_1, \ldots, g_k, f_k)$ is a sequence of vertices of $D_{T'}$ joined by edges, so if $\widetilde{f}$ is isolated in D_T and (f, g) is an edge of $D_{T'}$, then f is in the component K' and $D_{T'}$ is connected (up to isolated vertices).

(ii) Let $l_z > l_y$. Then the path form u to y is a sequence $(y_0 = u, x_1, y_1, \ldots x_{m-1}, y_{m-1} = z, x_m = y)$ with $(x_i, y_{i-1}) \in E(T')$ and (x_i, y_i) for $i = 1, \ldots, m$ (where $m = l_z/2 + 1$). It looks like this:



We will define k as in (i). If $k = m$, we will define g_1: $g_1(y_{m-1}) = y$, $g_1(w) = g(w)$ for $w \in V(T') \smallsetminus \{y_{m-1}\}$. The edge (y, y_{m-1}) is not source, so $(y, y_{m-1}) \notin A_{f^*}$ and the edge (f^*, g_1) exists. Vertex f_1 and vertices f_i, g_i are defined stepwise similarly as in (i), the only difference is that now we have $i = 2, \ldots, k-1$ and $f_{k-1} \in K'$. If $k < m$, we will define g_1 same as in (i) and again we have $(f^*, g_1) \in E(D_{T'})$. We will define f_i, g_i, but now, as in (i), we have $i = 2, \ldots, k$ and $f_k \in K'$. In any case we have found a path connecting f with some vertex that belongs to the set K'.

(2) There are no such vertices x,y,z in T'. Then if $\widetilde{f}$ is isolated in $D_{T'}$ and (f, g') is an edge of $D_{T'}$, then $f \in F$ and $g' \in G'$, where $F = \{f; f$ has no problematic sinks$\}$ and $G' = \{g'; g'$ has no problematic sources$\}$. Let $g' \in G'$. Let y be an arbitrary sink with $(u, x_1, y_1, \ldots, x_k, y)$ being the path from u to y in sym(T'). Let's suppose that g' maps y on some neighbor of the vertex y, say x_{k+1}, such that $x_{k+1} \neq x_k$. Since x_{k+1} is not problematic source (there are no problematic sources for g', according to hypothesis), it has some neighbor,

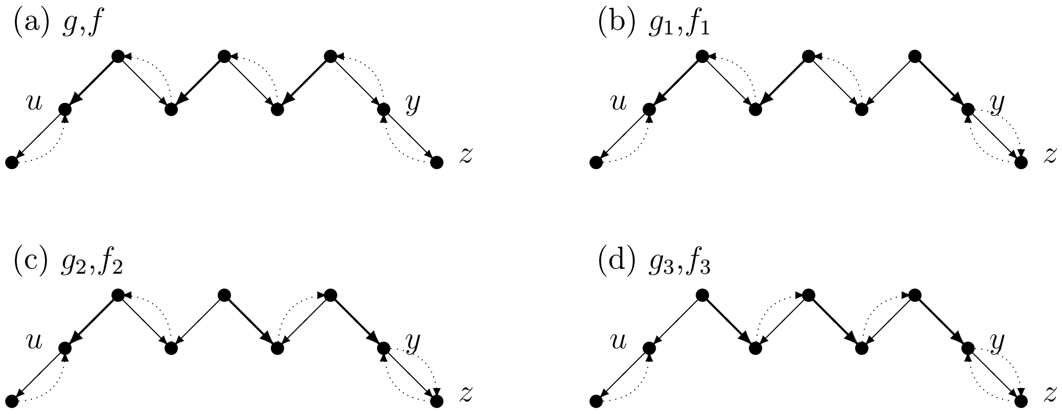


Figure 1: This is an illustration of the proof of the theorem 2, part (i). The vertices u and z are connected by exactly one path in $\mathrm{sym}(T')$. The subgraph of T' induced by its vertices can look (for example) like the one in the pictures. Picture (a) shows an example of mappings f,g, forming an edge (f, g). Edges in A_f are denoted by thicker lines. Dotted arrows show the edges belonging to B_g. Pictures (b), (c) and (d) show mappings g_i a f_i, which are constructed in the proof of the theorem. The edges from A_{f_i} are shown by thicker lines and the dotted arrows denote the direction of mappings g_i on the vertices of interest.

say y_{k+1}, such that $g'(y_{k+1}) = x_{k+2}$, where $x_{k+2} \neq x_{k+1}$. But neither x_{k+2} is a problematic source, thus it has a neighbor y_{k+2} such that $g'(y_{k+2}) = x_{k+3} \neq x_{k+2}$. This way we get a sequence $(u, x_1, y_1, \ldots, x_k, y, x_{k+1}, y_{k+1}, \ldots, y_{m-1}, x_m)$ for every integer m, where $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$. But that is a contradiction (out tree T' is finite). As a consequence, every $g' \in G'$ is uniquely determined on sinks of T': if y is a sink such that $y \neq v$ and $(u, x_1, y_1, \ldots, x_k, y)$ is a path from u to y in $\mathrm{sym}(T)$, then $g'(y) = x_k$. Moreover if $g'(u) = x \neq v$, then x is a problematic source, thus $g'(u) = v$. Obviously, if g' satisfies these conditions, then there exists no problematic source for it and $g' \in G'$; thus these conditions are not only necessary but also sufficient and all $g' \in G'$ map the same set B of edges against their direction.

Similarly, if $f \in F$, x is a source, $(u, x_1, x_2, \ldots, y_k, x)$ a path from u to x in $\mathrm{sym}(T)$ and $f(x) = y_{k+1} \neq x_k$, then since y_{k+1} is not a problematic sink, it has a neighbor x_{k+1} such that $f(x_{k+1}) = y_{k+2} \neq y_{k+1}$ etc.; thus we again get an infinite sequence in the finite tree T'. So if $f \in F$, x is a source and $(u, x_1, y_1, \ldots, y_k, x)$ is a path from u to x, then $f(x) = y_k$ and this condition is also sufficient (because $f(u) \neq v$). All $f \in F$ have the same set of edges that they map in their direction.

So if (f, g') is an edge, there exist also the edges (f_i, g') and (f, g'_i) for all $f_i \in F$ and $g'_i \in G'$. The symmetrization of $D_{T'}$ is then (after removing isolated vertices) a complete bipartite graph and $D_{T'}$ is (after removing isolated vertices) connected. $\square$

The case that the newly added edge is directed outwards from T is thus settled. But what if it has the opposite direction? We could prove lemmas and theorem equivalent to those above, but there is a simpler way:

Lemma 4 *Let T^{-1} denote an oriented tree obtained from T by reversing all edges. Then if D_T is connected after removing isolated vertices, then the same holds for $D_{T^{-1}}$.*

Proof. Obviously $V(D_T) = V(D_{T^{-1}})$. If (f,g) is an edge of D_T, then we have $A_f \cap B_g = \emptyset$, where $A_f = \{(u,v) \in E(T); f : u \mapsto v\}$ and $B_g = \{(u,v) \in E(T); g : v \mapsto u\}$. But $(u,v) \in A_f$ if and only if $(v,u) \in B_f^{-1}$ and $(u,v) \in B_g$ if and only if $(v,u) \in A_g^{-1}$, where $A_g^{-1} = \{(v,u) \in E(T^{-1}); g : v \mapsto u\}$ and $B_f^{-1} = \{(v,u) \in E(T^{-1}); f : u \mapsto v\}$. $A_f \cap B_g = \emptyset$ holds if and only if $A_g^{-1} \cap B_f^{-1} = \emptyset$, so (f,g) is an edge of D_T if and only if (g,f) is an edge of $D_{T^{-1}}$. □

Theorem 3 *Let T be an oriented tree and let $V(D_T) = \{\tilde{f}_1, \ldots, \tilde{f}_k\}$. Let T'' be defined as follows: $V(T'') = V(T) \cup \{u\}$ and $E(T'') = E(T) \cup \{(u,v)\}$. If D_T is connected after removing isolated vertices, then $D_{T''}$ is as well.*

Proof. We know that $D_{T^{-1}}$ is connected (except for isolated vertices) by previous lemma. Now we will define a tree T^*: $V(T^*) = V(T^{-1}) \cup \{u\}$ and $E(T^*) = E(T^{-1}) \cup \{(v,u)\}$ (vertex v given by the hypothesis). D_{T^*} is connected after removing isolated vertices (theorem 2). By lemma 4, $D_{(T^*)^{-1}}$ is also connected (except for isolated vertices). But $(T^*)^{-1} = T''$, thus we get the connectedness of $D_{T''}$. □

Now we have everything we need for proving the main theorem:

Theorem 4 *For any oriented tree T, D_T defined in proposition 1 is connected after removing isolated vertices.*

Proof. We proceed by induction on the number of edges of T. The theorem certainly holds for $T = K_2$ and theorems 2 and 3 provide the induction step. □

The proof of polynomial diameter uses the same idea as the proof of connectedness. We use the fact that after making T' from T by adding a leaf, a copy of D_T is a subgraph of $D_{T'}$ and we bound the length of paths not contained in this copy by some integer a. We obtain the following induction step: $\mathrm{diam}(D_{T'}) \leq \mathrm{diam}(D_T) + 2a$. To simplify notation in the next proof, I will use $\mathrm{diam}(D_T)$ as a shorthand for diam(sym("D_T with isolated vertices removed")).

Theorem 5 *Let T be an oriented tree with $n \geq 4$ vertices and D_T its dual defined in proposition 1. Then the following holds:*

$$diam(D_T) \leq n^2 - n - 8.$$

Proof. Again, we will call T' a tree that we obtain from T by adding an edge (of arbitrary direction). If D_T has some non-trivial connected component K, we will define a set $L = \{f; \tilde{f} \in K\}$ (I am using notation from definition 2) and let K' be the non-trivial connected component of $D_{T'}$. We have $L \subseteq K'$ by lemma 3 and for $f, g \in K'$ we have $d(f, g) \leq d(f, L) + \mathrm{diam}(L) + d(g, L)$ where $d(f, g)$ is the distance of f and g in the symmetrization of $D_{T'}$, $d(f, L) = \min\{d(f, h); h \in L\}$ and $\mathrm{diam}(L)$ is the diameter of (the symmetrization of) the subgraph of $D_{T'}$ induced by the set L. Using lemma 3 we obtain $\mathrm{diam}(L) = \mathrm{diam}(K) = \mathrm{diam}(D_T)$. We also have $d(f, L) = 0$ for $f \in L$, so all we need to do is to find a such that $d(f, L) \leq a$ for all $f \in K' \setminus L$. Then $\mathrm{diam}(D_{T'}) \leq \mathrm{diam}(D_T) + 2a$. What values can a have? To answer this question we will again distinguish a few cases.

(i) When making T' from T, we add a vertex v and an edge (u, v) to some vertex $u \in V(T)$, which is not sink. If $\tilde{f}$ is isolated, then so is f (since u is not sink). Thus from $h \in K' \setminus L$ we get $h \in \{f'_1, \dots, f'_k\}$, so there exists some $f \in \{f_1, \dots, f_k\}$ such that (f, h) is an edge. But that can happen only if $f \in L$, thus we obtain $a = 1$. If we add the edge (u, v) to a vertex v that is not source, the situation is analogous and again $a = 1$.

(ii) We add an edge (u, v) to a vertex u, which is a sink but which has some neighbor that is not a source. If $\tilde{f}$ is an isolated vertex but f is not, then we will define f^* the same way as in the proof of theorem 2. If (f, g) is an edge for some $g \in \{f'_1, \dots, f'_k\}$, then (f^*, g) is also an edge and $f^* \in L$. So we have $d(f, L) \leq 2$ for $f \in \{f_1, \dots, f_k\}$. For every $g \in \{f'_1, \dots, f'_k\} \cap K'$ there is an $f \in \{f_1, \dots, f_k\}$ such that (f, g) is an edge, and thus $d(g, L) \leq d(g, f) + d(f, L) \leq 3$, which gives us $a = 3$. Again, the situation is the same if we add an edge (u, v) to a source v that has a neighbor that is not a sink.

(iii) Again we add an edge (u, v) to a sink u, but this time all its neighbors are sources. If $P_2 \not\to T$, then D_T contains only isolated vertices and the symmetrization of $D_{T'}$ is (after removing isolated vertices) complete bipartite graph. Its diameter is at most 2, so by adding this edge we have increased the diameter by at most 2. The same holds if we add (u, v) to a vertex v of a tree T such that $P_2 \not\to T$.

(iv) We add (u, v) to a sink u whose all neighbors are sources, but now $P_2 \to T$. Again, let $\tilde{f}$ be an isolated vertex of D_T such that f is not isolated. As in the proof of theorem 2, we find numbers l_y, l_z and k and a sequence $(f, g, f^*, g_1, \dots, g_k, f_k)$. If $l_y < l_z$, then (f, g_1) is also an edge of $D_{T'}$, and $(f, g_1, f_1, \dots, g_k, f_k)$ is a path from f to f_k shorter than the above. We get $d(f, L) \leq 2k$ for $f \in \{f_1, \dots, f_k\}$ and $k \leq m = l_y/2$, so $d(f, L) \leq l_y$. If $g \in \{f'_1, \dots, f'_k\} \cap K'$, then $d(g, L) \leq d(g, f) + d(f, L) \leq l_y + 1$ (where $f \in \{f_1, \dots, f_k\}$ is a neighbor of g), thus $a = l_y + 1$. If $l_z < l_y$ and $k < m$, then again $(f, g_1) \in E(D_{T'})$ and we get a path $(f, g_1, f_1, \dots, g_k, f_k)$ of length $2k$ with endpoints f and $f_k \in L$. $2k \leq l_y - 1$ (because $k < m$), so $d(g, L) \leq d(g, f) + d(f, L) \leq l_y$ for $g \in \{f'_1, \dots, f'_k\}$ and its neighbor $f \in \{f_1, \dots, f_k\}$, and $a = l_y$. If $k = m$, we can't use the same reasoning. However, also in this case the length of the shortest path from f to L

is not longer than $2k$, because it ends with a vertex $f_{k-1} \in L$; it is a sequence $(f, g, f^*, g_1, \dots, g_{k-1}, f_{k-1})$. If (f, g) is an edge for $g \in \{f'_1, \dots, f'_k\}$, then (f^*, g) is also an edge, so $d(g, L) \leq d(g, f^*) + d(f^*, L) \leq 2k - 1$, where f is a neighbor of g. We get $a = l_y + 1$. Thus for all these cases we have an upper bound $a = l_y + 1$. Again, if we add (u, v) to a source v, whose all neighbors are sinks and $P_2 \to T$, the situation is analogous and we have $a = l_y + 1$.

Let T_n be a tree with n vertices. For small n we know $\operatorname{diam}(D_{T_n})$:

$$\begin{aligned}
n = 1 \ \dots \ & \operatorname{diam}(D_{T_1}) = 0 \\
n = 2 \ \dots \ & \operatorname{diam}(D_{T_2}) = 0 \\
n = 3 \ \dots \ & \operatorname{diam}(D_{T_3}) \leq 1 \\
n = 4 \ \dots \ & \operatorname{diam}(D_{T_4}) \leq V(D_{T_4}) \leq 4
\end{aligned}$$

As I already mentioned, $\operatorname{diam}(D_{T_{n+1}}) \leq \operatorname{diam}(D_{T_n}) + 2a$, where $a = 1$, $a = 3$ or $a = l_y + 1$. If $a = l_y + 1$, then $l_y \leq n - 1$ (because y has neighbors x and z and the distance of one of them from the place where we are adding the new edge is greater than l_y), so $a \leq n$. If $n \geq 3$, then $a \leq n$ holds also for $a = 3$ and $a = 1$ and the increase of diameter is bounded by the number $2n$. Thus for $n \geq 4$ we get $\operatorname{diam}(D_{T_n}) \leq \operatorname{diam}(D_{T_4}) + 8 + 10 + \dots + 2(n-1) \leq 4 + 8 + 10 + \dots + 2(n-1) = 2(1 + 2 + \dots + (n-1)) - 8 = 2 \cdot \frac{(n-1)n}{2} - 8 = n^2 - n - 8$. $\square$