An Introduction to Complex Random Matrices

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Abstract

We present an introduction to random matrices, and compute the eigenvalue distribution of an ensemble of complex $N \times N$ random matrices. We go on to compute the n $(1 \leq n \leq N)$ eigenvalue correlation functions. The eigenvalue distribution and correlation functions are obtained using the methods of Ginibre- however a more "hands on" description of his methods are presented here. Next, we derive the Gibbs canonical measure of N point charges at inverse temperature β and at positions $\lambda_1, \dots, \lambda_N \in \Lambda \subset \mathbb{R}^2$ and show how this distribution is *identical* to the eigenvalue distribution, $\mu^{(N)}$, of complex $N \times N$ random matrices for $\beta = 2$. Lastly, a discussion of possible avenues of progress (to be pursued this summer at the Mathematics REU at Rutgers University) will be presented.

1 INTRODUCTION

From the point of view of a newcomer to the field of Random Matrices it seems unnatural to speak of something being random other than some variable. But interestingly enough one can easily think of a hat filled with pieces of paper on which, say, 3×3 matrices have been written- lets say you are at a party (for fun math people) and each guest writes down a 3×3 matrix and places it in a hat without letting anyone else look at it. The hat would thus be an ensemble (collection), though small, of random 3×3 matrices! This can be done for any arbitrary $N \times N$ matrix with any type of entries: the entries could be complex, real, 1 or 0, etc. What we will seek to understand here are random $N \times N$ matrices with complex entries - whose eigenvalues we try to describe according to the methods of Ginibre.

Specifically, we will describe the eigenvalue density, $d\mu^{(N)}$, typically called the measure *on* or eigenvalue probability distribution *of* complex $N \times N$ matrices. In addition we will compute the eigenvalue distribution, sometimes called the joint probability distribution of eigenvalues, $\mu^{(N)}$ of complex $N \times N$ random matrices and show that this is *identical* to the Gibbs canonical measure of N point charges at inverse temperature $\beta = 2$ and at positions $\lambda_1, \ldots, \lambda_N \in \Lambda \subset \mathbb{R}^2$. Due to this astonishing similarity it is then possible to understand random matrices from the point of view of mathematical statistical physics - which seems to be under utilized as opposed to purely algebraic methods tried thus far. Several avenues for possible study from this mathematical physics side of the coin will then be presented.

2 Understanding Jean Ginibre's Paper:*Statistical Ensembles of Complex, Quarternion, and Real Matrices*

To understand the derivation of the eigenvalue distribution of complex $N \times N$ random matrices we turn to the 1964 paper of Jean Ginibre[1]. In what follows an intensive discussion of the methods of Ginibre is presented so that newcomers can quickly understand how random matrices are analyzed according to his methods. The discussion will be limited in scope to just complex random matrices; there will not be a discussion of quarternionic matrices nor real symmetric matrices. It is hoped that Ginibre's paper and this brief manuscript can easily acquaint newcomers to the rigors of the field.

2.1 Preliminaries: Ensembles and Measures

The first step in understanding random matrices is to define what exactly we mean by an *ensemble* of random matrices. This definition has two parts: 1.) an algebraic set of matrices, and 2.) a probability measure on this algebraic set. For the complex $N \times N$ matrices let us define the ensemble and probability measure as follows:

Definition (i.) *Let $\mathcal{Z}_N$ be the set of $N \times N$ complex matrices with statistically independent entries , for $S \in \mathcal{Z}_N$ we measure S by (or say that the eigenvalues of S have density) :*

$$d\mu^{(N)}(S) = d\mu_L(S)\exp[-\frac{1}{4a^2}TrS^\dagger S] \tag{1}$$

where: $d\mu_L(S)$ is the linear measure defined by:

$$d\mu_L(S) = \prod_{i,j} dS_{ij}dS^*_{ij} \tag{2}$$

where for $S_{ij} = x_{ij} + iy_{ij}, dS_{ij}dS^*_{ij} \equiv 2dxdy$.

To understand this definition let us consider the following examples:

Example 1: Linear Measure

[1] Ginibre, J., *Statistical ensembles of complex, quarternion, and real matrices,* J. Math. Phys. **6**, pp. 440-449 [1964]

For: $S_{ij} = x_{ij} + iy_{ij}$, one would expect that

$dS_{ij}dS^*_{ij} = (dx_{ij} + idy_{ij})(dx_{ij} - idy_{ij}) = dx^2_{ij} + dy^2_{ij}$. But this is not the case!

$dS_{ij}dS^*_{ij} = (dx_{ij} + idy_{ij})(dx_{ij} - idy_{ij})$ is *defined* to be equal to[2] $2dx_{ij}dy_{ij}$. $\square$

<u>Example 2:</u> Eigenvalue Probability Distribution of Matrices in $\mathcal{Z}_N$

Let: $\mathcal{Z}_3$ be the set of complex 3×3 matrices and $S \in \mathcal{Z}_3$. Suppose S is defined as:

$$S = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} + i \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \tag{3}$$

then after keeping in mind **Definition (i.)** and *Example 1* and performing a little algebra, one can see:

$$\begin{aligned}
d\mu^{(3)}(S) &= d\mu_L(S)\exp[-\frac{1}{4a^2}TrS^\dagger S] && (4)\\
&= \prod_{i,j\,=\,1}^{3} dS_{ij}dS^*_{ij}\exp[-\frac{1}{4a^2}\sum_{i,j\,=\,1}^{3}(x^2_{ij} + y^2_{i,j})] && (5)\\
&= dx_{11}dy_{11}dx_{12}dy_{12}\ldots dx_{32}dy_{32}dx_{33}dy_{33}\exp[-\frac{1}{4a^2}(x^2_{11} + \cdots && (6)\\
&\qquad \cdots + x^2_{32} + x^2_{33} + y^2_{11} + \cdots + y^2_{32} + y^2_{33})] && (7)\\
&= \prod_{i,j\,=\,1}^{3} dx_{ij}\exp[-\frac{x^2_{ij}}{4a^2}]\prod_{i,j\,=\,1}^{3} dy_{ij}\exp[-\frac{y^2_{ij}}{4a^2}] && (8)
\end{aligned}$$

Thus the probability measure[3], on $\mathcal{Z}_3$ is simply a product of Gaussian distributions for each of the real and imaginary parts of the entries of S multiplied by the uniform measure

[2]Note that the factor of 2 is commonly left out of the notation and is brought in at special times (which, at the time of writing, I have yet to see!). If this factor is needed it will be explained!

[3]To avoid confusion, which I had, let us recall that: the probability density $\rho(x)$ of a variable x and the probability distribution $P(x)$ for x have the following relationship: $\rho(x) = \frac{dP(x)}{dx}$. Thus what Ginibre refers to as the probability distribution, or measure, in his paper, is referred to here as a probability measure. That is, the probability measure will be integrated to obtain the probability distribution. Hence: $d\mu^{(N)}(S) \approx \rho(x)dx = dP(x)$. This will also make more sense when the probability distribution of the eigenvalues is considered, it will end up being a (marginal) integral of $d\mu^{(N)}(S)$, which makes sense as: $P(x) = \int dP(x)$.

on $Im(S_{ij})$ and $Re(S_{ij})$, ie. dy_{ij}, dx_{ij} respectively. $\square$

<u>Remark:</u> *From Example 2 it follows that we may write the measure given in (1) as*[4]:

$$d\mu^{(N)}(S) = \prod_{i,j} dx_{ij} \exp[-\frac{x_{ij}^2}{4a^2}] \prod_{i,j} dy_{ij} \exp[-\frac{y_{ij}^2}{4a^2}] \tag{9}$$

With our definition of an ensemble of complex $N \times N$ random matrices, coupled with an understanding of the measure on this set: $d\mu^{(N)}$ given by (9), our preliminary work is complete. We now turn to the calculation of the eigenvalue distribution $\mu^{(N)}$.

2.2 Calculation of $\mu^{(N)}$

The goal of this section will be to come up with a way in which to integrate $d\mu^{(N)}(S)$ of equation (9) so as to obtain $\mu^{(N)}(S)$, the eigenvalue distribution of the complex $N \times N$ matrices. That is we seek to compute something like:

$$\mu^{(N)} = \int d\mu^{(N)}(S) \tag{10}$$

To begin with let us take $4a^2$ in equation (1) to be 1. Then the eigenvalue distribution is defined as follows[5]:

Definition (ii.) *Let $S \in \mathcal{Z}_N$, and let $\lambda_i, i = 1, \ldots, N$, be the eigenvalues of S. Then*

$$\mu^{(N)}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N) \prod_{i=1}^{N} d\lambda_i d\lambda_i^* = mrg. \int d\mu^{(N)}(S) \tag{11}$$

Note: that $mrg. \int$ is a marginal integral in which for fixed λ_i we integrate over all other variables[6].

[4] Where each entry of S is represented by $S_{ij} = x_{ij} + iy_{ij}$.

[5] See Ginibre's paper, pg 441.

[6] See Appendix A for an example of a marginal integral.

Our efforts to integrate (11) will have a two part focus consisting firstly of expressing the linear measure in (1) for some generic element $S \in \mathcal{Z}_N$, i.e. $\prod_{i,j} dS_{ij} dS^*_{ij}$, as a measure on the eigenvalues of S. And secondly, doing such will reduce the marginal integral in (1) to the integration of a factor J yet to be defined. These two efforts will then allow a closed form expression for $\mu^{(N)}$ that will bare uncanny resemblance to the Gibbs canonical distribution of point charges in $\mathbb{R}^2$ at inverse temperature $\beta = 2$.

2.2.1 Focus 1: Working with the Linear Measure

Claim (i.):

We consider only complex $N \times N$ matrices with distinct eigenvalues, as matrices with two identical eigenvalues form a set of measure zero[7].

Implications of Claim (i.):

We may express any generic element S of $\mathcal{Z}_N$ as $S = XAX^{-1}$ where A is diagonal in the eigenvalues of S and X is some invertible matrix determined by usual methods. The upshot here is that we can now begin to consider the linear measure $d\mu^{(N)}$ on S as a product of entries of X, A, and X^{-1}, see (14).

Claim (ii.):

$$dS = X(dA + [X^{-1}dX, A])X^{-1}$$

Proof of Claim (ii.):

The relation presented in *Claim* (*ii.*) can be seen with some algebra. Consider $S = XAX^{-1}$ and $I = XX^{-1}$ then applying the product rule for differentiation:

$$dI = 0 \quad = \quad (dX)X^{-1} + X(dX^{-1}) \tag{12}$$

$$dS \quad = \quad dX(AX^{-1}) + X(dA)X^{-1} + (XA)dX^{-1} \tag{13}$$

[7]See Appendix A: This claim needs to backed up with some sort of example.

Solving (12) for dX^{-1} and substituting into (13), we obtain:

$$\begin{aligned} dS &= dX(AX^{-1}) + X(dA)X^{-1} + (XA)dX^{-1} \\ &= dX(AX^{-1}) + X(dA)X^{-1} + XA(-X^{-1}(dX)X^{-1}) \\ &= X(dA + X^{-1}dXA - AX^{-1}dX)X^{-1} \\ &= X(dA + [X^{-1}dX, A])X^{-1} \end{aligned}$$

as desired.

Implications of Claim (ii.):

Let us return to the linear measure given in (2), only now we substitute $dS_{ij} = (X(dA + [X^{-1}dX, A])X^{-1})_{ij}$.

$$d\mu_L(S) = d\mu_L(S = XAX^{-1}) = \prod_{i,j} \left(X(dA + [X^{-1}dX, A])X^{-1}\right)_{ij} \left(X(dA + [X^{-1}dX, A])X^{-1}\right)^*_{ij} \tag{14}$$

Thus the eigenvalues of S now enter into the linear measure, though in a complicated form with the involvement of $A = diag(\lambda_i, \ldots, \lambda_N)$ in the commutators. With $Claim(iii.)$ the eigenvalues will now be separated from their present cryptic involvement in (14).

Claim (iii.):

$$d\mu_L(S) = \prod_i d\lambda_i d\lambda_i^* \prod_{i \neq j} [X^{-1}dX, A]_{ij} [X^{-1}dX, A]^*_{ij}$$

Verifying Claim (iii.):

Claim (iii.) follows from the invariance property of the linear measure[8]: for some generic element $S \in \mathcal{Z}_N$, $d\mu_L(S) = d\mu_L(XSX^{-1})$ for any invertible X. Thus let $F \in \mathcal{Z}_N$ be defined such that $d\mu_L(F) = dA + [X^{-1}dX, A]$, then $d\mu_L(XFX^{-1}) = X(dA + [X^{-1}dX, A])X^{-1}$. Applying the invariance condition of the linear measure on XFX^{-1} one obtains: $d\mu_L(F) = \prod_{i,j}(dA + [X^{-1}dX, A])_{ij}(dA + [X^{-1}dX, A])^*_{ij}$. One can then recognize that $[X^{-1}dX, A]_{ij} = 0$ when $i = j$, to see this consider a 2×2 case:

let: $A = diag(\lambda_1, \lambda_2)$, and for $z_{ij} \in \mathbb{C}^1$:

$$
\begin{aligned}
X &= \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \\
X^{-1} &= \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{pmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{pmatrix} \\
dX &= \begin{pmatrix} dz_{11} & dz_{12} \\ dz_{21} & dz_{22} \end{pmatrix}
\end{aligned}
$$

then:

$$
\begin{aligned}
[X^{-1}dX, A] &= \begin{pmatrix} \lambda_1 z_{22}dz_{11} - \lambda_1 z_{12}dz_{21} & \lambda_2 z_{22}dz_{12} - \lambda_2 z_{12}dz_{22} \\ -\lambda_1 z_{12}dz_1 1 + \lambda_1 z_{11}dz_{21} & -\lambda_2 z_{21}dz_{12} + \lambda_2 z_{11}dz_{22} \end{pmatrix} \\
&\qquad - \begin{pmatrix} \lambda_1 z_{22}dz_{11} - \lambda_1 z_{12}dz_{21} & \lambda_1 z_{22}dz_{12} - \lambda_1 z_{12}dz_{22} \\ -\lambda_2 z_{21}dz_{11} + \lambda_2 z_{11}dz_{21} & -\lambda_2 z_{21}dz_{12} + \lambda_2 z_{11}dz_{22} \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\lambda_1 - \lambda_2)(z_{12}dz_{22} - z_{22}dz_{12}) \\ (\lambda_1 - \lambda_2)(z_{11}dz_{21} - z_{21}dz_{11}) & 0 \end{pmatrix}
\end{aligned}
$$

Now if we add in dA to obtain the full form of $dA + [X^{-1}dX, A]$,

$$
\mathrm{dA} + [\mathrm{X}^{-1}dX, A] = \begin{pmatrix} d\lambda_1 & (\lambda_1 - \lambda_2)(z_{12}dz_{22} - z_{22}dz_{12}) \\ (\lambda_1 - \lambda_2)(z_{11}dz_{21} - z_{21}dz_{11}) & d\lambda_2 \end{pmatrix} \tag{14}
$$

and consider the linear measure on the resulting 2×2matrix:

$$
\begin{aligned}
\prod_{i,j}(dA + [X^{-1}dX, A])_{ij}(dA + [X^{-1}dX, A])^*_{ij} &= d\lambda_1 d\lambda_1^* d\lambda_2 d\lambda_2^* \prod_{i \neq j = 1}^{2} [X^{-1}dX, A]_{ij}[X^{-1}dX, A]^*_{ij} \\
&= \prod_{i=1}^{2} d\lambda_i d\lambda_i^* \prod_{i \neq j = 1}^{2} [X^{-1}dX, A]_{ij}[X^{-1}dX, A]^*_{ij}
\end{aligned}
$$

[8] See appendix A.

Claim (iii.) is verified in the 2×2 case.

Claim (iv.):

$$d\mu_L(S) = \prod_i d\lambda_i d\lambda_i^* \prod_{i<j} |\lambda_i - \lambda_j|^4 \prod_{i \neq j} (X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij} \tag{15}$$

Verifying Claim (iv.) in the 2×2 case:

To verify *Claim (iv.)* we need only take the equation for the linear measure given by: $\prod_i d\lambda_i d\lambda_i^* \prod_{i\neq j} [X^{-1}dX, A]_{ij}[X^{-1}dX, A]^*_{ij}$ and substitute the appropriate values for the $i \neq j$ terms of $[X^{-1}dX, A]_{ij}$ and $[X^{-1}dX, A]^*_{ij}$ given in (15):

$$\begin{aligned} d\mu_L(dA + [X^{-1}dX, A]) &= d\lambda_1 d\lambda_1^* d\lambda_2 d\lambda_2^* ((\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2)^*)^2 (d(z_{12}dz_{22} - z_{22}dz_{12}) \times \\ &\qquad \times d(z_{12}dz_{22} - z_{22}dz_{12})^*) d(z_{11}dz_{21} - z_{21}dz_{11}) d(z_{11}dz_{21} - z_{21}dz_{11})^* \\ &= \prod_{i=1}^{2} d\lambda_i d\lambda_i^* \prod_{i<j=1}^{2} |\lambda_i - \lambda_j|^4 \prod_{i\neq j=1}^{2} (X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij} \end{aligned}$$

Implications of Claim (iv.):

Returning to the definition of the eigenvalue distribution given in (11), we can substitute our new form of the linear measure which now contains a measure on the eigenvalues of elements of $\mathcal{Z}_N$:

$$\begin{aligned} \mu^{(N)}(\lambda_1, \ldots, \lambda_N) \prod_{i=1}^{N} d\lambda_i d\lambda_i^* &= mrg. \int d\mu^{(N)}(S) \\ &= mrg. \int \prod_i d\lambda_i d\lambda_i^* \prod_{i<j} |\lambda_i - \lambda_j|^4 \prod_{i\neq j} (X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij} \exp[-TrS^\dagger S] \end{aligned}$$

thus:

$$\mu^{(N)}(\lambda_1, \ldots, \lambda_N) = \frac{1}{N!} \prod_{i<j} |\lambda_i - \lambda_j|^4 J \tag{16}$$

where:

$$J = \int \prod_{i\neq j} (X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij} \exp\left[-TrS^\dagger S\right] \tag{17}$$

Focus 1 is now complete. We were able to rewrite the linear measure (2) as a measure on eigenvalues (16) that we substituted into (11). The resulting integral defined by J is exactly the "integral around the margin" we are told to compute in **Definition (ii.)** for the fixed eigenvalues, λ_i, of the generic element $S \in \mathcal{Z}_N$. That is, it is the integral over the non-eigenvalue elements of $S = XAX^{-1}$.

We turn now to Ginibre's factorization argument for any $S \in \mathcal{Z}_N$ with distinct eigenvalues so as to compute J. We will find that the computation of (18) will allow a closed form solution for $\mu^{(N)}$ that is identical to the Gibbs canonical measure of point charges in two dimensions at inverse temperature $\beta = 2$.

2.2.2 Focus 2: Towards the Integration of J

In order to compute (18) we need to in some way get the integral into a more workable form that we can apply usual integration methods to, specifically we want something of the form:

$$J = \int \prod_{i,j} d\xi_{ij} d\xi^*_{ij} \exp(-\sum_i |\xi|^2_i) \tag{18}$$

This aim will have two aspects: working on the measure $\prod_{i \neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$ and the argument $S^\dagger S$ of the exponential in (18) so as to get the integration variables to correspond as they do in (19). As it stands, the $Tr[S^\dagger S]$ in (18) has some component of X as $S = XAX^{-1}$, but the specific correspondence with the measure $\prod_{i \neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$ is not exactly clear.

Ginibre addresses this by proving that X can be written uniquely as $X = UYV$, where U is unitary, Y is (upper)triangular, and V is diagonal with real positive elements. This allows for a different representation of the measure $\prod_{i \neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$ after proving some general facts that follow from the new representation of X. Ginibre then goes on to express the $S^\dagger S$ term in the trace in terms of the new representation of X.

At the end of the day the expressions for the new measure and $S^\dagger S$ still don't match up, however a simple change of variables allows for a final integrable form of J to be found.

We first present the claims that Ginibre proves in his paper and see how these allow for a new representation of the measure. We then go on to consider the impact of writing $X = UYV$ on the trace of $S^\dagger S$. Then the last change of variables will be performed and finally the integration of J will be carried out so that we can return to the determination of the eigenvalue distribution $\mu^{(N)}$.

2.2.3 Preliminaries: The Representation $X = UYV$ and Its Consequences[9]

We state the results of Ginibre without proof, however their meaning will be evident in their use throughout the rest of the development.

Claim (v.):

Let $d\mu_o(X) = \prod_{i,j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$. Then we say $d\mu_o(X)$ is the invariant measure on the group $GL(N, \mathbb{C}^1)$ and $\prod_{i\neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$ is the product measure on $GL(N, \mathbb{C}^1)/a'$ (which is not a group), where a' is the set of all complex diagonal matrices.

Claim (vi.):

Any $X \in GL(N, \mathbb{C}^1)$ has a unique representation of $X = UYV$, where U is unitary, Y is triangular ($Y_{ij} = 0$ for $i > j$ and $Y_{ii} = 1$ for $i = 1, \ldots, N$) and V is diagonal with real positive elements.

Claim (vii.):

[9]See Ginibre, pg 441.

The invariant measure $d\mu_o(X) = \prod_{i,j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$ can be written as $d\mu_o(X) = d\mu_o(\mathcal{U})d\mu_o(\mathcal{Y})d\mu_o(\mathcal{V})$ where $d\mu_o(\mathcal{U})$ is the invariant measure on the set $\mathcal{U}$ of unitary matrices, $d\mu_o(\mathcal{Y})$ is the invariant measure on the set $\mathcal{Y}$ of triangular matrices, and $d\mu_o(\mathcal{V})$ is the invariant measure on the set $\mathcal{V}$ of diagonal matrices with real positive elements. Thus $d\mu_o(GL(N,\mathbb{C}^1)) = d\mu_o(\mathcal{U})d\mu_o(\mathcal{Y})d\mu_o(\mathcal{V})$.

Claim (viii.):

The invariant measure on the set a' of complex diagonal matrices is given by $d\mu_o(a') = d\mu_o(\mathcal{U}_o)d\mu_o(\mathcal{V})$, where $\mathcal{U}_o$ is the set of diagonal unitary matrices, and $\mathcal{V}$ is the set of diagonal matrices with real positive elements.

2.2.4 Working with the measure: $\prod_{i\neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$

Claim (*v.*) tells us that $\prod_{i\neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij} = d\mu_o(GL(N,\mathbb{C}^1))/d\mu_o(a')$ then from *Claim* (*vii.*) and *Claim* (*viii.*) we may say:

$$\begin{aligned}
\prod_{i\neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij} &= \frac{d\mu_o(GL(N,\mathbb{C}^1))}{d\mu_o(a')} \\
&= \frac{d\mu_o(\mathcal{U})d\mu_o(\mathcal{Y})d\mu_o(\mathcal{V})}{d\mu_o(\mathcal{U}_o)d\mu_o(\mathcal{V})} \\
&= \frac{d\mu_o(\mathcal{U})d\mu_o(\mathcal{Y})}{d\mu_o(\mathcal{U}_o)}
\end{aligned}$$

Now we can substitute the new representation of the measure $\prod_{i\neq j}(X^{-1}dX)_{ij}(X^{-1}dX)^*_{ij}$ into (18) to obtain:

$$J = \int \frac{d\mu_o(\mathcal{U})d\mu_o(\mathcal{Y})}{d\mu_o(\mathcal{U}_o)} \exp\left[-TrS^\dagger S\right] \tag{19}$$

The integral of the invariant measure on $\mathcal{U}_o$ can be computed[10] and is equal to $(2\pi)^N$, as well let us write $\Omega_u = \int d\mu_o(\mathcal{U})$, and keeping in mind that
$d\mu_o(\mathcal{Y}) = \prod_{i<j}(Y^{-1}dY)_{ij}(Y^{-1}dY)^*_{ij}$ (20) becomes:

$$J = \frac{\Omega_u}{(2\pi)^N}\int d\mu_o(\mathcal{Y})\exp\left[-TrS^\dagger S\right] = \frac{\Omega_u}{(2\pi)^N}\int\prod_{i<j}(Y^{-1}dY)_{ij}(Y^{-1}dY)^*_{ij}\exp\left[-TrS^\dagger S\right] \tag{20}$$

2.2.5 Working with the Trace of $S^\dagger S$:

Let us begin by simply expanding the trace of $S^\dagger S$, first by writing $S = XAX^{-1}$, then substituting $X = UYV$.

$$\begin{aligned}
Tr[S^\dagger S] &= [(XAX^{-1})^\dagger(XAX^{-1})] \\
&= Tr[\big((UYV)A(UYV)^{-1}\big)^\dagger\big((UYV)A(UYV)^{-1}\big)] \\
&= Tr[\big(UYVAV^{-1}Y^{-1}U^{-1}\big)^\dagger\big(UYVAV^{-1}Y^{-1}U^{-1}\big)] \\
&= Tr[\big((U^{-1})^\dagger(Y^{-1})^\dagger(V^{-1})^\dagger A^\dagger V^\dagger Y^\dagger U^\dagger\big)\big(UYVAV^{-1}Y^{-1}U^{-1}\big)]
\end{aligned}$$

Four pieces of information will allow us to reduce this expression: 1.) $U^\dagger = U^{-1}$, that is U is unitary. 2.)Because V has real positive elements, $(V^{-1})^T = V^{-1}$ as well $V^T = V$. 3.) A and V as well as A and V^{-1} commute as they are both diagonal matrices. 4.) The Trace is cyclically invariant, that is $Tr[ABC] = Tr[BCA] = Tr[CAB]$.

$$\begin{aligned}
Tr[S^\dagger S] &= Tr[A^\dagger(V^{-1})^\dagger V^\dagger Y^\dagger U^\dagger UYAVV^{-1}Y^{-1}U^{-1}(U^{-1})^\dagger(Y^{-1})^\dagger] \\
&= Tr[A^\dagger Y^\dagger YAY^{-1}(Y^{-1})^\dagger]
\end{aligned}$$

[10] See Weyl

Then we note that $(Y^{-1})^\dagger = (Y^\dagger)^{-1}$, thus:

$$Tr[S^\dagger S] \;=\; Tr[AY^\dagger Y A(Y^\dagger Y)^{-1}] \tag{21}$$

Substituting (22)into (21):

$$J = \frac{\Omega_u}{(2\pi)^N} \int \prod_{i<j} (Y^{-1}dY)_{ij}(Y^{-1}dY)^*_{ij} \exp\left(-Tr[AY^\dagger Y A(Y^\dagger Y)^{-1}]\right) \tag{22}$$

Our work with J is now almost complete. Our last task before integrating is to let $Y^\dagger Y = H$ and change variables from $Y^{-1}dY$ to dH.

2.2.6 Changing Variables[11] from $Y^{-1}dY$ to dH.

Changing variables from $(Y^{-1}dY)_{ij}$ to dH_{ij} will be a two step process. We will first transform from $(Y^{-1}dY)_{ij}$ to dY_{ij}, then from dY_{ij} to dH_{ij}.

Going from $(Y^{-1}dY)_{ij}$ to dY_{ij}:

Let $det(\Lambda)$ be the Jacobian of the transformation from $(Y^{-1}dY)_{ij}$ to dY_{ij}. That is, from elementary calculus, $\prod_{i<j}(Y^{-1}dY)_{ij}(Y^{-1}dY)^*_{ij} = \prod_{i<j}(dY)_{ij}dY)^*_{ij}det(\Lambda)$, where Λ is the matrix:

$$\Lambda = \begin{pmatrix} 1 & \frac{\partial(Y^{-1}dY)_{12}}{\partial dY_{12}} & \frac{\partial(Y^{-1}dY)_{13}}{\partial dY_{13}} & \cdots & \frac{\partial(Y^{-1}dY)_{1N}}{\partial dY_{1N}} \\ 0 & 1 & \frac{\partial(Y^{-1}dY)_{23}}{\partial dY_{23}} & \cdots & \frac{\partial(Y^{-1}dY)_{2N}}{\partial dY_{2N}} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{\partial(Y^{-1}dY)_{N-1,N}}{\partial dY_{N-1,N}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

[11] See Ginibre, pg. 442.

Expanding in cofactors of the last row it is evident that the determinant of Λ is simply 1, thus:

$$\prod_{i<j}(Y^{-1}dY)_{ij}(Y^{-1}dY)^*_{ij} = \prod_{i<j}(dY)_{ij}dY)^*_{ij}det(\Lambda) = \prod_{i<j}(dY)_{ij}dY)^*_{ij}$$

<u>*Going from dY_{ij} to dH_{ij}:*</u> As has been done several times throughout this paper, we will be forced to simply state Ginibre's results and then verify them through example. Let us now do this for the transformation from dY_{ij} to dH_{ij}.

<u>*Claim (ix.)*</u>

$$\prod_{i<j} dY_{ij}dY^*_{ij} = \prod_{i<j} dH_{ij}dH^*_{ij}$$

where:

$$H_{ij} = \begin{cases} Y_{ij} + \sum_{k<i} Y^*_{ki}Y_{kj} & (i<j) \\ Y^*_{ji} + \sum_{k<j} Y^*_{ki}Y_{kj} & (i>j) \end{cases}$$

<u>*Verifying Claim (ix.)*</u>

Consider the following 3×3 example, let:

$$Y = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$\prod_{i<j=1}^{3} dY_{ij}dY^*_{ij} = dz_{12}dz^*_{12}dz_{13}dz^*_{12}dz_{23}dz^*_{23}$$

Now lets take a look at $Y^\dagger Y$:

$$H = Y^\dagger Y = \begin{pmatrix} 1 & 0 & 0 \\ z^*_{12} & 1 & 0 \\ z^*_{13} & z^*_{23} & 1 \end{pmatrix} \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_{12} & z_{13} \\ z^*_{12} & z_{12}z^*_{12}+1 & z^*_{12}z_{13}+z_{23} \\ z^*_{13} & z^*_{13}z_{12}+z^*_{23} & z^*_{13}z_{13}+z^*_{23}z_{23}+1 \end{pmatrix}$$

Then[12]:

[12] My interpretation of $d(z^*_{12}z_{13}+z_{23})d(z^*_{12}z_{13}+z_{23})^*$ is that it is $= dz_{12}z^*_{12}z_{13}z^*_{13}z_{23}z^*_{23}$, as well I take $(dz_{ij}dz^*_{ij})^2$ to be $= dz_{ij}dz^*_{ij}$.

$$\prod_{i<j=1}^{3} dH_{ij}dH_{ij}^* = dz_{12}dz_{12}^*dz_{13}dz_{13}^*d(z_{12}^*z_{13}+z_{23})d(z_{12}^*z_{13}+z_{23})^*$$
$$= dz_{12}dz_{12}^*dz_{13}dz_{13}^*dz_{23}dz_{23}^*$$

Claim (x.)

$$\prod_{i<j} dH_{ij}dH_{ij}^* = \prod_{i\neq j} dH_{ij}$$

Verifying Claim (x.)

Returning to the 3×3 case considered for $Claim$ $(ix.)$ one can easily verify $Claim$ $(x.)$:

$$H == \begin{pmatrix} 1 & z_{12} & z_{13} \\ z_{12}^* & z_{12}z_{12}^*+1 & z_{12}^*z_{13}+z_{23} \\ z_{13}^* & z_{13}^*z_{12}+z_{23}^* & z_{13}^*z_{13}+z_{23}^*z_{23}+1 \end{pmatrix}$$

then:

$$\prod_{i\neq j=1}^{3} dH_{ij} = dz_{12}dz_{12}^*dz_{13}dz_{13}^*d(z_{12}^*z_{13}+z_{23})d(z_{12}^*z_{13}+z_{23})^*$$
$$= dz_{12}dz_{12}^*dz_{13}dz_{13}^*dz_{23}dz_{23}^*$$

With the change of variables now complete, J takes the following integrable form:

$$J = \frac{\Omega_u}{(2\pi)^N}\int\prod_{i\neq j} dH_{ij}\exp\left(-Tr[AHAH^{-1}]\right) \tag{24}$$

2.2.7 Focus 2: The Integration of J and the Final Form of $\mu^{(N)}$

2.2.8 The n-Eigenvalue Correlation Functions for $\mu^{(N)}$

3 Derivation of the Gibbs Canonical Distribution

Throughout this report on the developments put forth by Ginibre there has been mention of the "Gibbs Canonical Distribution of N point charges at positions $\lambda_1, \ldots, \lambda_N \in \mathbb{R}^2$ at inverse temperature $\beta = 2$". It is this expression that bares uncanny resemblance to the eigenvalue distribution we arrived at in the previous section. What follows is a brief introduction to the language of mathematical statistical physics and the statistical description of point charges in two dimensions. This section will culminate with the derivation of the Gibbs canonical measure of point charges in two dimensions. A discussion of how the mathematical statistical physics point of view of point charges in two dimensions developed here can be applied to understanding eigenvalues of random $N \times N$ complex matrices will then be presented.

3.0.9 Point Charges in Two Dimensions

This section closely follows the instructive set of lectures by Minlos[13] which are apart of the University Lecture Series books put out by the American Mathematical Society.

The Classical Gas

Let $\Lambda \subset \mathbb{R}^2$ be a bounded domain. Consider N identical point particles moving around inside Λ. Each particle has a position $q_i \in \Lambda$ and a velocity $v_i \in \mathbb{R}^2, i = 1, \ldots, N$. The state of this system of particles moving around in Λ is defined by the knowledge of each particles position and velocity; that is the sequence: $((q_1, v_1), (q_2, v_2), \ldots, (q_N, v_N)) \equiv (Q, V)$. Q can be thought of as the collection of positions of particles in Λ while V can be thought of the corresponding set of velocities associated with each particle.

The space $\Lambda \times \mathbb{R}^2$ is the space of states (*phase space*) of a single particle- that is $\Lambda \times \mathbb{R}^2$ possess all the possible positions and velocities of a single particle in Λ. For a collection

[13]Minlos, R.A., *Introduction to Mathematical Statistical Physics*, University Lecture Series Vol. 19,*AMS*, Providence, R.I. [2000]

of N particles that comprise our system we can define the space of states of the entire N particles as : $\Omega_{\Lambda,N} = ((\Lambda \times \mathbb{R}^2) \times (\Lambda \times \mathbb{R}^2) \times \cdots \times (\Lambda \times \mathbb{R}^2)) \subset (\mathbb{R}^2 \times \mathbb{R}^2)^N = \mathbb{R}^{4N})$.

The energy of the system is expressed as a function of the state (Q, V) of the system. This function is most commonly called the Hamiltonian and it take the general form:

$$H_{\Lambda,N}(Q,V) = \sum_{i=1}^{N} \frac{mv_i^2}{2} + \sum_{1 \le i < j \le N} U_{int}(q_i - q_j) + \sum_{i=1}^{N} U_{ext}(q_i) \tag{25}$$

The first sum in $H_{\Lambda,N}$ represents the kinetic energy while the second and third sums represent the potential energy. m is the mass of each of the identical particles and $U_{int}(q_i - q_j)$ is the particle-particle interaction and $U_{ext}(q_i)$ is the interaction of the particle with an external force such as a magnetic field.

The particles move inside Λ according to the following system of differential equations (here they are the classical Newton equations) called the equations of motion for each particle labeled by i:

$$m\frac{dv_i}{dt} = \sum_{i \ne j} F_{int}(q_i - q_j) + F_{ext}(q_i) \tag{26}$$

$\frac{dq_i}{dt}$ defines the velocity, v_i, of each of the particles in Λ. F_{int} represents the force exerted on particle i by j, F_{ext} represents the force exerted on a particle i by an external force to the system. F_{int} and F_{ext} are the gradients of the scalar potentials U_{int} and U_{ext} respectively, ie: $F_{int} = -\nabla U_{int}$ and $F_{ext} = -\nabla U_{ext}$. In general it is supposed that the repulsive force F_{int} becomes infinitely negative (repulsive) as $r \to 0$ as $U_{int} \to \infty$ as $r \to 0$. Moreover it is supposed that the interaction force between particles acts only at close distances and that there is a radius of interaction beyond which particles do not influence one another.

A next logical step would be to discuss the dynamics of the two-dimensional system- but as this concept and those related to it are not pertinent to the discussion at hand we refer the reader to Minlos, pages 5 and 6.

The Configuration Gas

The configuration gas is very similar to the classical gas, however the configurational gas leaves out a description of the velocities of the particles in Λ. This is done as a simplification when specific properties of the system do not rely on the velocities of each contingent particle.

Under these circumstances the state of the system is given by the *configuration* of its contingent particles, the sequence $Q = (q_1, \ldots, q_N), q_i \in \Lambda, i = 1, \ldots, N, \Lambda \subset \mathbb{R}^2$. The phase space of the system is the Cartesian product: $\Omega^{conf}_{\Lambda,N} \equiv \Lambda \times \cdots \times \Lambda \subset \mathbb{R}^{2N}$. The energy is again represented as a function of the state of the system, this time only in terms of the configuration (coordinates) of the particles:

$$H_{\Lambda,N}(Q) = \sum_{1 \le i < j \le N} U_{int}(q_i - q_j) + \sum_{i=1}^{N} U_{ext}(q_i) \tag{27}$$

Both U_{int} and U_{ext} retain their previous descriptions.

Gibbs Canonical Ensemble (Distribution)

Now in speaking of a system in which there are many particles equipped with their own detailed motions our goal will not be to describe every last particle of the system. Instead our goal will be to describe the certain fraction of the particles within the system possessing a given property. That is we will be after a description of the collective behavior of a set of particles whose individual motions and states are non essential to this understanding. An example of this could be a description of particles with a certain spin or energy.

Mathematically we will express these collective properties in terms of probabilities of events defined in terms of a probability distribution P on the phase space, $\Omega_{\Lambda,N}$, of the system. The up shot of such a description is that for any physically observable property that can be represented as a function $F(\omega)$, $\omega \in \Omega_{\Lambda,N}$, of the state of the system, we can find its mean value with respect to the distribution P: $\langle F\rangle_P = \int_{\Omega_{\Lambda,N}} F(\omega)dP(\omega)$. As an example we could find the mean value of the energy of the system $\langle H\rangle_P$ or the mean value of the total velocity $\langle\sum_i v_i\rangle_P$.

Working within the limit that our system of particles in two dimensions are at thermal equilibrium, our goal now is to come up with just such a probability measure P that will allow for a realistic description of our system: we consider the Gibbs canonical measure on the classical gas and then go on to consider the same measure for the configurational gas. The measure applied to the configurational gas will lead us to the eigenvalue distribution found in section 2.2.7.

The Gibbs canonical distribution, $\mu^{canon}_{\Lambda,N,\beta}$ for $\beta > 0$ is defined as:

$$\mu^{canon}_{\Lambda,N,\beta} = \int_{\Omega_{\Lambda,N}} d\mu^{canon}_{\Lambda,N,\beta}(Q,V) = \int_{\Omega_{\Lambda,N}} \frac{\exp(-\beta H_{\Lambda,N}(Q,V))}{Z_{\Lambda,N,\beta}} \prod_{i=1}^{N} dq_i \prod_{i=1}^{N} dv_i \qquad (28)$$

Here β is the inverse temperature parameter and $Z_{\Lambda,N,\beta}$ is the normalizing factor:

$$Z_{\Lambda,N,\beta} = \int_{\Omega_{\Lambda,N}} e^{-\beta H_{\Lambda,N}(Q,V)} \prod_{i=1}^{N} dq_i \prod_{i=1}^{N} dv_i \qquad (29)$$

known as the partition function. To obtain the Gibbs canonical distribution for the configurational ensemble we consider the configurational Hamiltonian of (27) and substitute it into (28), as well we use the two dimensional Coulomb repulsion potential for U_{int} in (27) and consider a quadratic potential.

The Coulomb Potential in Two Dimensions

Our goal is to determine the form of U_{int} for point particles with unit charge in $\mathbb{R}^2$. Noting that in two dimensions[14] the electric field of a point charge is given by:

$$\vec{E} = \frac{\hat{r}}{r}$$

The potential, U_{int}, is related to $\vec{E}$ by:

$$\vec{\nabla} \cdot \vec{E}r = \nabla^2 U_{int}$$

Then by usual methods, U_{int} takes the form:

$$U_{int} = -\ln(|r_i - r_j|)$$

Let us now return to (28) and consider the density $d\mu_{\Lambda,N}^{conf}$:

$$d\mu_{\Lambda,N}^{conf} = Z_{\Lambda,N,\beta}^{-1} \exp(-\beta H_{\Lambda,N}(Q)) \prod_{i=1}^{N} dq_i$$

with the (configurational) Hamiltonian $H_{\Lambda,N}$ comprised of a logarithmic particle-particle interaction potential and a quadratic external potential :

$$\begin{aligned} H_{\Lambda,N}(Q) &= \sum_{1 \le i < j \le N} U_{int}(q_i - q_j) + \sum_{i=1}^{N} U_{ext}(q_i) \\ &= \sum_{1 \le i < j \le N} -\ln(|q_i - q_j|) + \sum_{i=1}^{N} -q_i^2 \end{aligned}$$

Upon substitution of $H_{\Lambda,N}$ into $d\mu_{\Lambda,N}^{conf}$:

[14]Griffiths, David, J., *Introduction to Electrodynamics*, Prentice Hall, New Jersey [1999], pg.$_?$

$$d\mu_{\Lambda,N}^{conf} = Z_{\Lambda,N,\beta}^{-1} \prod_{1\leq i<j\leq N} |q_i - q_j|^\beta \prod_{i=1}^{N} \exp(-q_i^2) dq_i \tag{30}$$

Thus in developing the configurational canonical measure for point charges in two dimensions, we arrive at the very expression that (using $\beta = 2$)represents the eigenvalue density of the random $N \times N$ complex matrices.

4 Future Plans: Mathematical Statistical Physics and Random Matrices

Project 1.

Computation of pair distribution functions of charges in two dimensions.

Project 2.

Compute $-Tr[S^\dagger S]$ for things like $S^\dagger S S^\dagger S$.

Project 3.

Explore the Quantum Field Theoretic approaches to complex Random matrices.