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On the bounds for the ultimate independence ratio of a graph

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Abstract

In this paper we study the ultimate independence ratio $I(G)$ of a graph G, which is defined as the limit of the sequence of independence ratios of powers of G. We construct a graph G with ultimate independence ratio $I(G)$ strictly between the previous known upper bound $1/\chi_f(G)$ and lower bound $1/\chi(G)$. We give a better lower bound for $I(G)$ by using the concept of star-chromatic number of a graph.

1. Introduction

For a graph G, the independence ratio $i(G)$ is defined as $i(G) = \alpha(G)/|V(G)|$, where $\alpha(G)$ is the maximum cardinality of an independent set of G. The Cartesian product $G \square H$ of graphs G and H has vertex set $V(G \times H) = \{(g, h): g \in V(G), h \in (H)\}$ and two vertices (g, h) and (g', h') are adjacent, written $(g, h) \sim (g', h')$, if either $g = g'$ and $h \sim h'$ or $g \sim g'$ and $h = h'$. The power G^k is the Cartesian product of k copies of G.

The ultimate independence ratio $I(G)$ of a graph G, defined as $I(G) = \lim_{k \to \infty} i(G^k)$, was first introduced and studied in [8]. It is easy to see that $i(G^k)$, $k = 1, 2, \ldots,$ is a non-increasing sequence, and that $i(G^k) \geqslant 1/\chi(G^k) = 1/\chi(G)$ (cf. [8]). Therefore the limit exists and $i(G) \geqslant I(G) \geqslant 1/\chi(G)$.

A better upper bound for $I(G)$ is given in [6] by studying the homomorphism of graphs. A *homomorphism* of a graph G to a graph H is a mapping $f: V(G) \to V(H)$ such that $f(u) \sim f(v)$ whenever $u \sim v$. We say G is *homomorphic* to H, and write $G \to H$, if there exists a homomorphism of G to H. It was proved in [6] (Theorem 2.1) that if there is a homomorphism of G to H, then $I(H) \leqslant I(G)$, which implies that $I(G) \leqslant 1/\chi_f(G)$ for any graph G (Theorem 3.1). Here $\chi_f(G)$ is the *functional chromatic number* of G, which is defined as the minimum total weight that can be distributed on

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the independent sets of G so that the sum of the weights of independent sets covering a vertex is at least one for every vertex of G.

There is no known algorithm to determine $I(G)$. The few graphs of which the value $I(G)$ has been determined are all special graphs. For example, it has been determined for Cayley graphs of Abelian groups, the Petersen graph, etc. [1, 2, 14]. For these graphs, $I(G)$ is always equal to either the upper bound $1/\chi_f(G)$ or the lower bound $1/\chi(G)$. This raised the question whether this is always the case, i.e. whether $I(G)$ is always equal to $1/\chi_f(G)$ or $1/\chi(G)$. More generally it was asked in [6] whether or not $I(G)$ is always rational. In this paper a graph G with $I(G)$ strictly between this upper bound and lower bound is constructed. Then we give a better lower bound for $I(G)$ by using the concept of star-chromatic number of a graph.

2. An example

In this section, we use the wreath product of graphs to construct a graph G for which the ultimate independence ratio $I(G)$ is strictly between the bounds $1/\chi_f(G)$ and $1/\chi(G)$.

Let G and H be graphs. The *wreath product* of G and H, denoted by $G[H]$, has vertex set $V(G[H]) = \{(g,h)\colon\ g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G[H]) = \{((g,h)),(g',h'))\colon$ either $(g,g') \in E(G)$ or $g = g'$ and $(h,h'') \in E(H)\}$. (The wreath product is also known as the lexicographic product.)

Lemma 1. *For any graphs G and H, we have $I(G[H]) \geqslant I(G)I(H)$.*

Proof. It suffices to show that $i((G[H])^k) \geqslant i(G^k)\cdot i(H^k)$. Since $|V((G[H])^k)| = |V(G^k)|\cdot|V(H^k)|$, we only need to show that $\alpha((G[H])^k) \geqslant \alpha(G^k)\cdot\alpha(H^k)$. Let S be an independent set of G^k and let T be an independent set of H^k. We shall show that the set $R = \{((a_1,b_1),\ (a_2,b_2),\ldots,(a_k,b_k))\colon\ (a_1,a_2,\ldots,a_k) \in S,\ (b_1,b_2,\ldots,b_k) \in T\}$ is an independent set of $(G[H])^k$. For otherwise suppose $v = ((a_1,b_1), (a_2,b_2),\ldots,(a_k,b_k))$ and $u = ((a'_1,b'_1), (a'_2,b'_2),\ldots,(a'_k,b'_k)$ are vertices in R, and that $v \sim u$ in $(G[H])^k$. Then by definition there is an index $1 \leqslant j \leqslant k$ such that $(a_i,b_i) = (a'_i,b'_i)$ for all $i \neq j$, and $(a_j,b_j) \sim (a'_j,b'_3)$ in $G[H]$. Thus we have either $a_j \sim a'_j$, which contradicts the assumption that $(a_1,a_2,\ldots,a_k)$ are not adjacent to $(a'_1,a'_2,\ldots,a'_k)$ in G^k, or $b_j \sim b'_j$, which contradicts the assumption that $(b_1,b_2,\ldots,b_k)$ is not adjacent to $(b'_1\ b'_2,\ldots,b'_k)$ in H^k. Therefore R is an independent set of $(G[H])^k$. Since $|R| = |S|\cdot|T|$, we have $\alpha((G[H])^k) \geqslant \alpha(G^k)\cdot\alpha(H^k)$, and hence $I(G[H]) \geqslant I(G)\cdot I(H)$. □

Theorem 1. *Let P be the Petersen graph, and let C_5 be the cycle of length 5. Then the graph $G = C_5[P]$ satisfies $1/\chi_f(G) > I(G) > 1/\chi(G)$.*

Proof. It is easy to verify that $\chi(G) = 8$. Since $I(P) = \frac{1}{3}$ (cf. [1]), and $I(C_5) = \frac{2}{5}$ (cf. [8]), we have $I(G) \geqslant \frac{2}{15}$ by Lemma 1. Therefore $I(G) > 1/\chi(G)$.

To prove that $I(G) < i(G)$, we use the idea employed in [8]. We define the *independence graph* of G, denoted by $\mathrm{Ind}(G)$, to be the graph with vertex set all the maximum independent sets of G and two maximum independent sets are adjacent in $\mathrm{Ind}(G)$ if they are disjoint as subsets of $V(G)$. It is easy to see that each maximum independent set of $G = C_5[P]$ corresponds to a maximum independent set of C_5, namely a maximum independent set of G contains 8 vertices, 4 of them from one copy of the Petersen graph and the other 4 vertices from another copy of the Petersen graph and the two copies of the Petersen graph correspond to two non-adjacent vertices of C_5. Any two maximum independent sets of G are disjoint if and only if the corresponding maximum independent sets of C_5 are disjoint. Therefore $\mathrm{Ind}(G)$ is homomorphic to $\mathrm{Ind}(C_5)$.

However $\mathrm{Ind}(C_5)$ is isomorphic to C_5. Therefore any graph homomorphic to $\mathrm{Ind}(G)$ is homomorphic to C_5 by composition of homomorphisms. It is well known, and also easy to see, that if a graph H is homomorphic to a graph H' then $\chi(H) \leqslant \chi(H')$. Thus any graph homomorphic to $\mathrm{Ind}(G)$ has chromatic number at most three. Therefore G is not homomorphic to $\mathrm{Ind}(G)$. As observed in [8], this implies that $i(G^2) < i(G)$. Indeed if $i(G^2) = i(G)$, and S is a maximum independent set of G^2, then the mapping $f\colon V(G) \to 2^{V(G)}$ defined as $f(x) = \{y \in V(G)\colon (x, y) \in S\}$ must be a homomorphism of G to $\mathrm{Ind}(G)$. Therefore $I(G) \leqslant i(G^2) < i(G)$. Since G is a vertex transitive, we have $1/\chi_f(G) = i(G)$ (cf. [6]). Therefore $I(G)$ is strictly between the upper bound $1/\chi_f(G) = i(G) = \frac{4}{25}$ and the lower bound $1/\chi(G) = \frac{1}{8}$. $\square$

A similar argument will show that for any $k \geqslant 3$, the graph $C_{2k+1}[P]$ also has ultimate independence ratio strictly between the above upper bound and lower bound.

3. A better lower bound

The study of the ultimate independence ratio can be viewed in the spirit of investigating the limiting behaviour of graph parameters under graph product. There are many other parameters whose limiting behaviour with respect to various products has attracted attention, cf. [7, 9–11]. We now consider the limiting behaviour of the fractional chromatic number of a graph, and relate it to the ultimate independence ratio of a graph. This relation is then used to derive a better lower bound for the ultimate independence ratio of a graph.

We define the *ultimate fractional chromatic number* $\chi_F(G)$ of a graph G as $\chi_F(G) = \lim_{k\to\infty} \chi_f(G^k)$.

Theorem 2. *The equality $I(G) \cdot \chi_F(G) = 1$ holds for all graphs G.*

Proof. It was proved in [6] (Theorem 3.1) that for any graph G we have $I(G) \leqslant 1/\chi_f(G)$. From the definition of the fractional chromatic number, it is easy to see that $1/\chi_f(G) \leqslant i(G)$. Since $I(G^k) = I(G)$, we have

$$I(G) = I(G)^k \leqslant \frac{1}{\chi_f(G^k)} \leqslant i(G^k).$$

Therefore $\lim_{k \to \infty} 1/\chi_f(G^k) = I(G)$, i.e. $I(G) \cdot \chi_F(G) = 1$. □

Recently I learned that Theorem 2 was proved independently by Favaron [4]. In the following we use this relation and the notion of the star-chromatic number of a graph to derive a new lower bound for the ultimate independence ratio of a graph.

The star-chromatic number of a graph, a concept introduced by Vince [13], is a natural generalization of the chromatic number of a graph. Some basic properties of this new graph parameter can be found in [3, 13, 15].

Definition 1. Let k and d be positive integers such that $k \geqslant 2d$. A (k,d)-coding of a graph $G = (V, E)$ is a mapping $c: V \to Z_k = \{0, 1, \ldots, k-1\}$ such that for each edge $uv \in E$, $|c(u) - (c(v)|_k \geqslant d$, where $|x|_k = \min\{|x|, k - |x|\}$.

We denote by G_k^d the graph with vertex set $Z_k = \{0, 1, \ldots, k-1\}$ and edge set $\{(i,j): |i-j|_k \geqslant d\}$. Then a (k,d)-colouring of a graph G is simply a homomorphism of G to G_k^d.

Definition 2. The star-chromatic number $\chi^*(G)$ of a graph G is defined as

$$\chi^*(G) = \min\,[k/d : G \text{ has a } (k,d)\text{-colouring and } k \leqslant |V(G)|\}.$$

In order to give our new lower bound for $I(G)$, we introduce the concept of a *partition tree* of a graph.

Definition 3. Let G be a graph. A partition tree T of G is a rooted tree such that each vertex of T is a subset of the vertex set of G, and

- the root is the set $V(G)$,
- each leaf vertex of T contains a single vertex of $V(G)$,
- if $A \subset V(G)$ is a vertex of T which is not a leaf, then its sons, as subsets of $V(G)$, form a partition of the set A.

Let T be a partition tree of a graph G, and let A be a vertex of T. The level $\lambda(A)$ of A is the length of the path connecting A and the root. (For example the root has level zero.) The level $\lambda(T)$ of the tree T is the maximum level of vertex of T. We also associate with each vertex A of T the following graph G_A: the vertices f G_A are the sons of A, and two sons U and V are adjacent if and only if there is a vertex x of G in U and

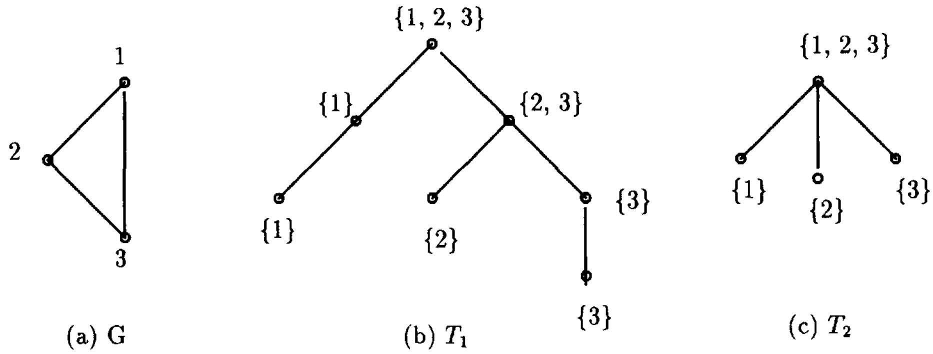

Fig. 1. A graph G and two partition trees of G.

a vertex y of G in V such that x and y are adjacent in G. The graph G_A can also be obtained as follows: Consider A as a subgraph of G, i.e. the subgraph induced by A. Then contract each of its sons into a single vertex. The resulting graph is G_A. If A is a leaf vertex, then G_A is a graph with a single vertex.

In Fig. 1, we illustrate two partition trees of the graph G in (a). The partition tree T_1 has level 3 and T_2 has level 1. The graph associated to the vertex $\{1,2,3\}$ in the tree T_1 is a copy of K_2, and the graph associated to the vertex $\{1,2.3\}$ in the tree T_2 is a copy of K_3.

Suppose the level of T *is* m. For each $0 \leqslant i \leqslant m$, we define

$$\alpha_T(G,i) = \max\{\chi^*(G_A) \colon A \in V(T),\ \lambda(A) = i\}.$$

We define $\alpha_T(G) = \alpha_T(G,0) \cdot \alpha_T(G,1) \cdot \cdots \cdot \alpha_T(G,m)$.

Let $\mathscr{T}(G)$ be the set of all partition trees of a graph G. We define

$$\chi^*_{\mathscr{T}}(G) = \min\{\alpha_T(G) \colon T \in \mathscr{T}(G)\}.$$

Observe that the set $\{\alpha_T(G) \colon T \in \mathscr{T}(G)\}$ is finite. Indeed suppose that T is a partition tree of G such that for some $i < \lambda(T)$, all vertices A of T with level i has a single son. Let T' be the tree obtained from T by contracting all the edges of T connecting the level i vertices and the level $i+1$ vertices, i.e. identify each of the level i vertices with its son. It is easy to see that T' is also a partition tree of G and $\alpha_T(G) = \alpha_{T'}(G)$. Thus to determine $\chi^*_T(G)$, it suffices to consider those partition trees T for which each level $i < \lambda(T)$ has a vertex with at least two sons. Obviously there are only finitely many such partition trees of G. Therefore the parameter $\chi^*_{\mathscr{T}}(G)$ is well defined. In fact for any graph $G, \chi^*_{\mathscr{T}}(G)$ can be determined by an obvious exhaustive algorithm.

Theorem 3. *For any graph G we have $I(G) \geqslant 1/\chi^*_{\mathscr{T}}(G)$.*

We first prove two lemmas.

Lemma 2. *For any graph G we have* $\chi_f(G) \leqslant \chi_{\mathcal{T}}^*(G)$.

Proof. Suppose T is a partition tree of G with $\lambda(T) = m$. We need to show that $\chi_f(G) \leqslant \alpha_T(G)$. Without loss of generality we can assume that each leaf vertex A of T has level $\lambda(A) = \lambda(T) = m$. Indeed if there is a leaf vertex A of T such that $\lambda(A) < \lambda(T)$, then we can attach to A a new vertex which is the same set A. The resulting tree T' is obviously a partition tree of G with $\alpha_{T'}(G) = \alpha_T(G)$.

Let $k_i/d_i = \alpha_T(G, i)$, $i = 0, 1, \ldots, m$, where k_i, d_i are integers. (As the star-chromatic number of any finite graph is rational, so $\alpha_T(G, i)$ is rational.) Let A be a vertex of T of level i. Since $\chi^*(G_A) \leqslant k_i/d_i$, the graph G_A is homomorphic to $G_{k_i}^{d_i}$ (cf. [3, 13]). Let f_A be a homomorphism of G_A to $G_{k_i}^{d_i}$, where $i = \lambda(A)$. For each vertex x of G, let $A_m(x)$, $A_{m-1}(x), \ldots, A_0(x)$ be the unique path of T connecting the leaf vertex $A_m(x) = \{x\}$ and the root A_0. Then it is easy to verify that the mapping f defined as $f(x) = (f_{A_0(x)}(A_1(x))$, $f_{A_1(x)}(A_2(x)), \ldots, f_{A_{m-1}}(x)(A_m(x)))$ is a homomorphism of G to the wreath product $H = G_{k_0}^{d_0}[G_{k_1}^{d_1}[\cdots[G_{k_m}^{d_m}]\cdots]]$. Therefore $\chi_f(G) \leqslant \chi_f(H)$, as any distribution of weights to the independent sets of H induces a distribution of weights to the independent sets of G of the same total weight via the homomorphism in an obvious way.

It remains to show that $\chi_f(H) \leqslant \alpha_T(G)$. The graph H is vertex transitive, thus $\chi_f(H) = 1/i(H)$. It is easy to see that $i(G_k^d) = d/k$, and $i\{A[B] = i(A) \cdot i(B)$ for general graph A and B. Thus $i(H) = d_0/k_0 \cdot d_1/k_1 \cdot \cdots \cdot d_m/k_m = 1/\alpha_T(G)$. Lemma 2 is proved. $\square$

Lemma 3. *For any graph G we have* $\chi_{\mathcal{T}}^*(G^k) = \chi_{\mathcal{T}}^*(G)$ *for all integers k.*

Proof. It is easy to see from the definition that if A is a subgraph of B then $\chi_{\mathcal{T}}^*(A) \leqslant \chi_{\mathcal{T}}^*(B)$. Thus $\chi_{\mathcal{T}}^*(G) \leqslant \chi_{\mathcal{T}}^*(G^k)$.

Suppose that T is a partition tree of G such that $\alpha_T(G) = \chi_{\mathcal{T}}^*(G)$. Without loss of generality, we assume that each leaf vertex A of T has level $\lambda(A) = \lambda(T)$. We construct a partition tree T^k of the graph G^k with $\lambda(T^k) = \lambda(T)$ as follows: The ith level has as vertices all the products $A_1 \times A_2 \times \cdots \times A_k$, where each A_i is a vertex of the ith level of T (note that A_i need not be distinct). A vertex $B_1 \times B_2 \times \cdots \times B_k$ on the $(i+1)$-level of T^k is a son of $A_1 \times A_2 \times \cdots \times A_k$ if and only if B_i is a son of A_i for each $1 \leqslant i \leqslant k$.

It is easy to see that T^k is a partition tree of G^k. Since for any two graphs A and B we have $\chi^*(A \square B) = \max\{\chi^*(A), \chi^*(B)\}$ (cf. [15]), and since $G_{A_1 \cdots A_k} = G_{A_1} \square \cdots \square G_{A_k}$, we have that $\alpha_{T^k}(G^k, i) = \alpha_T(G, i)$ and thus $\alpha_{T^k}(G^k) = \alpha_T(G) = \chi_{\mathcal{T}}^*(G)$. Therefore $\chi_{\mathcal{T}}^*(G^k) \leqslant \chi_{\mathcal{T}}^*(G)$ and hence $\chi_{\mathcal{T}}^*(G^k) = \chi_{\mathcal{T}}^*(G)$. $\square$

Proof of Theorem 3. By Theorem 2, it suffices to show that $\chi_f(G^k) \leqslant \chi_{\mathcal{T}}^*(G)$ for all integers k. This is true because $\chi_f(G^k) \leqslant \chi_{\mathcal{T}}^*(G^k)$ (Lemma 2) and $\chi_{\mathcal{T}}^*(G^k) = \chi_{\mathcal{T}}^*(G)$ (Lemma 3). $\square$

Corollary 1. *For any graph G we have* $I(G) \geqslant 1/\chi^*(G)$.

Proof. This corollary can be derived from Theorem 3 by considering the simplest partition tree T of G (i.e. the tree with the root adjacent to all the leaf vertices). An easy direct proof can be obtained by showing that $i(G) \geqslant 1/\chi^*(G)$ and $\chi^*(G^m) = \chi^*(G)$ for any integer m.

Since $\chi^*(G) \leqslant \chi(G)$ for all graphs G, and $\chi^*(G)$ is strictly less than $\chi(G)$ for most graphs, the bound in Corollary 1 is better than the previous known lower bound. We shall see in an example that the lower bound in Theorem 3 is strictly bigger than the bound of Corollary 1.

It was believed that if a graph G has a universal vertex, i.e. a vertex which is adjacent to every other vertex of G, then $I(G)$ should be equal to $1/\chi(G)$ (cf. [6]). In the following we construct a graph G which has a universal vertex and show that $I(G) > 1/\chi(G)$ by using the new lower bound of Theorem 3.

Let G be the graph obtained from K_{12} by deleting the edges of two vertex disjoint copies of C_5. This graph can also be obtained by taking the direct sum of two copies of C_5 and one copy of K_2. It is easy to see that G has a universal vertex (in fact G has two universal vertices) and that $\chi(G) = 8$. Let T be the partition tree of G as depicted in Fig. 2, where vertices $[1, 2]$ induce a copy of K_2, and each of the vertex sets $\{2, 3, \ldots, 7\}$ and $\{8, 9, \ldots, 12\}$ induces a copy of C_5. Then it is easy to see that $\alpha_T(G) = 3 \times 2.5 = 7.5$. Therefore $I(G) \geqslant 1/\chi^*_{\mathscr{T}}(G) = 1/7.5 > 1/\chi(G)$.

We note that if a graph G has a universal vertex then $\chi^*(G) = \chi(G)$ (Theorem 3 of [15]). Therefore we cannot prove $I(G) > 1/\chi(G)$ by using the bound of Corollary 1 for any graph G with a universal vertex.

An alternate proof of Theorem 3 can be given by showing that $H = G_{k_0}^{d_0}[G_{k_1}^{d_1}[\cdots[G_{k_m}^{d_m}]\cdots]]$ is a circulant graph. Then $G \to H$ implies that $I(G) \leqslant I(H) = i(H) = \alpha_{\mathrm{T}}(G)$ (the inequality holds by Theorem 2.1 of [6], the first equality holds because H is a circulant graph, the second equality was proved in the proof of Lemma 2). Such a proof shows that the lower bound for $I(G)$ in Theorem 3 is obtained by considering some particular Cayley graphs of Abelian groups which are homomorphic images of the graph G. From this point of view, it is natural to ask the

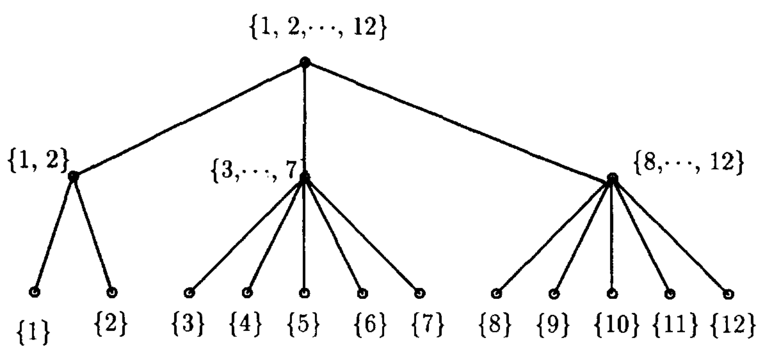

Fig. 2.

question whether or not we can get a better lower bound by considering all the Cayley graphs of Abelian groups which are homomorphic images of G. To be precise, we have the following inequality.

(*): $I(G) \geqslant \sup\{i(H): G \to H$, and H is a Cayley graph of an Abelian group$\}$.

The inequality (*) really gives a better lower bound for $I(G)$. There exist Cayley graphs H of Abelian groups for which $I(H) \neq \chi^*_{\mathscr{F}}(H)$. For example, the graph $H = K_5 \square K_2[C_5]$ is a Cayley graph of an Abelian group. It has ultimate independence ratio $I(H) = i(H) = 9/50$ and $\chi^*_{\mathscr{F}}(H) = 6$. Thus $I(H) > 1/\chi^*_{\mathscr{F}}(H)$. However, unlike the lower bound in Theorem 3, there is no known algorithm to determine the value of the right-hand side of the inequality (*) for general graphs.

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