The Shannon Capacity of Graphs and related topics

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Cvičení

Exercises

1. * Prove that K_{97} can be decomposed into 4 isomorphic, vertex and edge transitive graphs. What is the maximum clique size of this graph? Do the same except this time decompose it into 3 graphs.

2. * Let p be an odd prime number. Prove that $\theta(QR(p^2)) = \alpha(QR(p^2)) = p$.

3. Prove that $\vartheta(Cl) = 6$ where Cl is the Clebsch graph.

4. Let $V(G) = GF(16)$. Two vertices x, y are connected by an edge if $x + y = \beta^3$ for some $\beta \in GF(16)$. Find $\alpha(G), \vartheta(G)$. Can you identify this graph?

5. Prove that the Schläfli graph Sl is a $(16, 10, 8)$-strongly regular graph.

6. Construct an orthogonal representation for Sl with stem vector $\mathbf{b}$ such that $\mathbf{b}^2 = 3$.

7. a. Let G be a triangle-free, 2-connected cubic graph. Prove that $\frac{|G|}{3} \leq \theta(G) \leq \frac{|G|}{2}$.

   b. Prove that if $\theta(G) = \frac{|G|}{2}$ then G is bipartite.

8. Let $\mathbf{A(n) = \{(a_1, \ldots, a_{2n}) | a_i \in 0, 1, 2, 3, 4,\ a_{2i-1} < a_{2i}\ 1 \geq i \geq n\}}$. Let $\mathbf{B \subset A(n),\ |B| > (\frac{5}{2})^n}$. Prove that there are two vectors b and c in $\mathbf{B}$ such that $|\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\}| \leq 1\ \forall k$.

   *** Is $(\frac{5}{2})^n$ best possible?

9. Prove that $\theta(G) + K_n = \theta(G) + 1$.

10. Prove that if $\chi(\overline{G}) = \alpha(G)$ then $\theta(G + H) = \theta(G) + \theta(H)$.
    (This was already observed by Shannon in 1956).

11. Can you prove the similar claim: if $\alpha^*(G) = \alpha(G)$ then $\theta(G + H) = \theta(G) + \theta(H)$.

12. * Prove that $\forall m,\ 9 \leq m \leq 17$, there is a graph G_m with $\alpha(G_m) = 3$ and $\alpha(G_m^2) = m$.
13. Prove that $\lim_{n\to\infty} \theta(C_{2^n+1}) - \alpha^*(C_{2^n+1}) = 0$.
14. * Prove that $\lim_{n\to\infty} \theta(C_{2n+1}) - \alpha^*(C_{2n+1}) = 0$.
15. Construct a symmetric, PSD matrix B that satisfies the conditions of theorem 7.0.3 for the Petersen graph with $tr(B\mathbf{J}) = 4$.
16. ** Let $K(n,r)$ be the Kneser graph $(n > 2r)$. Let $\mathbf{u} = (u_1, \ldots, u_n) \in R^n$ be a vector such that $\sum_{i=1}^{n} u_i = 0$. For any r-subset $A \subset \{1, \ldots, n\}$ let $x_A = \sum_{i\in A} u_i$. Prove that the vector (x_A) is a characteristic vector of $K(n,r)$ and its corresponding eigenvalue is: -$\binom{n-r-1}{n-1}$.
17. * Prove that the multiplicity of the eigenvalue -$\binom{n-r-1}{n-1}$ is n.
18. ** Prove that the eigenvalue -$\binom{n-r-1}{n-1}$ is the smallest eigenvalue of $K(n,r)$ and hence $\alpha(K(n,r)) = \vartheta(K(n,r)) = \binom{n-1}{r-1}$.
19. Prove that for any graph G there is a matrix $\mathbf{A} = (a_{ij})$ such that $a_{ii} = 0,\ a_{ij} = 0$ when $(i,j) \in E(G)$ and such that $\vartheta(G) \leq 1 - \frac{\lambda_n(\mathbf{A})}{\lambda_1(\mathbf{A})}$.
20. Prove that $\lim_{n\to\infty} \theta(C_{2^n+1}) - \alpha^*(C_{2^n+1}) = 0$.
21. Prove that $\beta(Z_{23}^2) = 8$.
22. Prove that $\beta(Z_7^3) = 7$.
23. Construct an orthogonal representation for $\overline{P}$ with a stem vector b such that $||b||^2 = \frac{5}{2}$. (Hint: let the vectors have two coordinates: α, β. Choose them so they form an orthogonal representation and select an appropriate stem vector).
24. * The Kneser graphs $K_n(3m-1, m)$ are triangle free. Prove that $\theta(\overline{K_n(3m-1,m)})$ <3.1 (in other words, if you wish to improve the upper bound on the Ramsey number $R(3;k)$ via the strong product these graphs will not help you. Hint: construct an orthogonal representation with the stem vector $(a, a, \ldots, a)$ for a properly selected a).
25. Prove that $\theta(L(K_{2n})) = n$ ($L(G)$ is the line graph of G).
26. Let $C_5 \oplus I_2$ be the prism over C_5. Prove that $\theta(C_5 \oplus I_2) = \sqrt{20}$ (so this will be another example of a connected graph for which the optimal word length is 2).
27. Show that $\alpha(K_{1,n^2}) = \theta(K_{1,n^2}) > \frac{n\lambda_n}{\lambda_n - \lambda_1}$.

Hints to selected exercises