The Shannon Capacity of Graphs and related topics

Moshe Rosenfeld

12 December 1999

Lecture 1, První Přednáška

1

Course roadmap, background and basic definitions and notation

The goal of this course is to explore beautiful interactions between various branches of mathematics via the Shannon Capacity of graphs. The branches include Graph Theory, Combinatorics, Ramsey Theory, Geometry, Linear Algebra, Number Theory and more. We shall expose many open problems and thrive to either solve them or make some progress towards their solution.

The Capacity of Noisy Channels was introduced by C. Shannon in 1956 [**?**]. In a nutshell, it is yet "another" graph parameter but the theory around it, developed primarily by L. Lovász, is rich in depth beauty and power. In his introductory paper Shannon left some open problems. Some have been solved and others still wait to be solved, a fertile ground for attractive and chalenging mathematical discoveries and explorations.

I plan to include at the end of every lecture a list of problems. Many of the problems will be designed to prepare us for the coming lectures. Some of the problems will be routine problems, intended to help us get a feeling for the basic notions, some will be more challenging, exploring the use of tools from other areas to solve combinatorial problems, some will be more subtle, usually calling for a clever idea and many open problems. Non-routine problems (problems for which I happen to know a nice trick for solving them) will be identified by an *. The more challenging will get **. Exploratory problems (those which I do not know how to solve, but find attractive and interesting) will be identified by ***.

The road will start with a short description of Shannon's original paper, noisy channels, and some very basic properties of noisy channels. Once we review the Information Theory background we'll quickly "shed" it and stay within the beautiful world of combinatorics and Graph Theory. Our next exploration will take us to the interaction with other combinatorial topics such as Ramsey Numbers and more. We'll then embark on various representations of Graphs (algebraic and geometric) and explore their relation to the Shannon capacity and their own merit. Next we'll explore bounds of the Shannon Capacity and in particular the beautiful theory developde by L. Lovász. This will probably be interlaced with studying eigenvalues techniques. We then explore the power of this theory and (time permitting) other applications in Geometry of the tools developed in this class.

In this introductory chapter, we first explain some basic notions and techniques, mostly fundamental and simple ones common to most branches of mathematics. We assume that the reader is already familiar with many of them or has at least heard of them. Thus, we will mostly review the notions, give precise formal definitions, and point out various ways of capturing the meaning of these concepts by diagrams and pictures. We then introduce the notions of Noisy Channels, the Shannon Capacity and explore its basic properties.

Section 1.1 is a review of notation. It introduces some common symbols for operations with sets and numbers, such as $\cup$ for set union or $\sum$ for summation of a sequence of numbers. Most of the symbols are standard, and the reader should be able to go through this section fairly quickly, relying on the index to refresh memory later on.

In Section 1.2, we briefly discuss some basic notions and common notation from Graph Theory.

In section 1.3 we introduce the notions of noisy channels, their characteristic graphs and the zero error capacity of noisy channels. This measures the amount of information a channel can communicate. This study was initiated By Claude E. Shannon in 1956.

1.1 Numbers and sets: notation

Number domains. For the set of all natural numbers, i.e. the set $\{1, 2, 3, \ldots\}$, we reserve the symbol $\mathbf{N}$. The letters n, m, k, i, j, p and possibly some others usually represent natural numbers.

Using the natural numbers, we may construct other well-known number domains: the integers, the rationals, and the reals (and also the complex numbers, but we will seldom hear about them here).

The *integer numbers* or simply *integers* arise from the natural numbers by adding the negative integer numbers and 0. The set of all integers is denoted by **Z**.

The *rational numbers* are fractions with integer numerator and denominator. This set is usually denoted by **Q** but we need not introduce any symbol for it in this book. The construction of the set **R** of all *real numbers* is more complicated, and it is treated in introductory courses of mathematical analysis. Famous examples of real numbers which are not rational are numbers such as $\sqrt{2}$, some important constants like π, and generally numbers whose decimal notation has an infinite and aperiodic sequence of digits following the decimal point, such as $0.12112111211112\ldots$.

The *closed interval* from a to b on the real axis is denoted by $[a, b]$, and the *open interval* with the same endpoints is written as (a, b).

Operations with numbers. Most symbols for operations with numbers, such as $+$ for addition, $\sqrt{\ }$ for square root, and so on, are generally well known. We write *division* either as a fraction, or sometimes with a slash, i.e. either in the form $\frac{a}{b}$ or as a/b.

We introduce two less common functions. For a real number x, the symbol $\lfloor x \rfloor$ is called[1] the *lower integer part of x* (or the *floor function* of x), and its value is the largest integer smaller than or equal to x. Similarly $\lceil x \rceil$, the *upper integer part of x* (or the *ceiling function*), denotes the smallest integer greater than or equal to x. For instance, $\lfloor 0.999 \rfloor = 0$, $\lfloor -0.1 \rfloor = -1$, $\lceil 0.01 \rceil = 1$, $\lceil \frac{17}{3} \rceil = 6$, $\lfloor \sqrt{2} \rfloor = 1$.

Later on, we will introduce some more operations and functions for numbers, which have an important combinatorial meaning and which we will investigate in more detail. Examples are $n!$ and $\binom{n}{k}$.

Sums and products. If $a_1, a_2, \ldots, a_n$ are real numbers, their sum $a_1 + a_2 + \cdots + a_n$ can also be written using the *summation sign* $\sum$, in the form

$$\sum_{i=1}^{n} a_i.$$

This notation somewhat resembles the `FOR` loop in many programming languages. Here are a few more examples:

$$\sum_{j=2}^{5} \frac{1}{2j} = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}$$

[1] In the older literature, one often finds $[x]$ used for the same function.

$$\sum_{i=2}^{5} \frac{1}{2j} = \frac{1}{2j} + \frac{1}{2j} + \frac{1}{2j} + \frac{1}{2j} = \frac{2}{j}$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j) &= \sum_{i=1}^{n}\left((i+1)+(i+2)+\cdots+(i+n)\right)\\ &= \sum_{i=1}^{n}\left(ni+(1+2+\cdots+n)\right)\\ &= n\left(\sum_{i=1}^{n} i\right) + n(1+2+\cdots+n)\\ &= 2n(1+2+\cdots+n).\end{aligned}$$

Similarly as sums are written using $\sum$ (which is the capital Greek letter "sigma", from the word *s*um), products may be expressed using the sign $\prod$ (capital Greek "pi"). For example,

$$\prod_{i=1}^{n} \frac{i+1}{i} = \frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{n+1}{n} = n+1.$$

Sets. Another basic notion we will use is that of a set. Most likely you have already encountered sets in high school (and, thanks to the permanent modernization of the school system, maybe even in elementary school). Sets are usually denoted by capital letters:

$$A, B, \ldots, X, Y, \ldots, M, N, \ldots$$

and so on, and the elements of sets are mostly denoted by lowercase letters: $a, b, \ldots, x, y, \ldots, m, n, \ldots$.

The fact that a set X contains an element x is traditionally written using the symbol $\in$, which is a somewhat stylized Greek letter ε—"epsilon". The notation $x \in X$ is read "x is an element of X", "x belongs to X", "x is in X", and so on.

Let us remark that the concept of a set and the symbol $\in$ are so-called primitive notions. This means that we do not define them using other "simpler" notions (unlike the rational numbers, say, which are defined in terms of the integers). To understand the concept of a set, we rely on intuition (supported by numerous examples) in this book. It turned out at the beginning of the 20th century that if such an intuitive notion of a set is used completely freely, various strange situations, the so-called paradoxes, may arise.[2] In order to exclude such paradoxes, the theory of sets has been rebuilt on a formalized basis, where all properties of sets are derived formally from several precisely formulated basic assumptions (axioms). For the sets used in this text, which are mostly finite, we need not be afraid of any paradoxes, and so we can keep relying on the intuitive concept of a set.

The set with elements 1, 37, and 55 is written as $\{1, 37, 55\}$. This, and also the notations $\{37, 1, 55\}$ and $\{1, 37, 1, 55, 55, 1\}$, express the same thing. Thus, a multiple occurrence of the same element is ignored: the same element cannot be contained twice in the same set! Three dots (an ellipsis) in $\{2, 4, 6, 8, \ldots\}$ mean "and further similarly, using the same pattern", i.e. this notation means the set of all even natural numbers. The appropriate pattern should be apparent at first sight. For instance, $\{2^1, 2^2, 2^3, \ldots\}$ is easily understandable as the set of all powers of 2, while $\{2, 4, 8, \ldots\}$ may be less clear.

Ordered and unordered pairs. The symbol $\{x, y\}$ denotes the set containing exactly the elements x and y, as we already know. In this particular case, the set $\{x, y\}$ is sometimes called the *unordered pair* of x and y. Let us recall that $\{x, y\}$ is the same as $\{y, x\}$, and if $x = y$, then $\{x, y\}$ is a 1-element set.

We also introduce the notation (x, y) for the *ordered pair* of x and y. For this construct, the order of the elements x and y is important. We thus assume the following:

$$(x, y) = (z, t) \text{ if and only if } x = z \text{ and } y = t. \tag{1.1}$$

Interestingly, the ordered pair can be defined using the notion of unordered pair, as follows:

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Verify that ordered pairs defined in this way satisfy the condition (1.1). However, in this text it will be simpler for us to consider (x, y) as another primitive notion.

Similarly, we write $(x_1, x_2, \ldots, x_n)$ for the *ordered n-tuple* consisting of elements $x_1, x_2, \ldots, x_n$. A particular case of this convention is

[2]The most famous one is probably Russell's paradox. One possible formulation is about an army barber. An army barber is supposed to shave all soldiers who do not shave themselves—should he, as one of the soldiers, shave himself or not? This paradox can be translated into a rigorous mathematical language and it implies the inconsistency of notions like "the set of all sets".

writing a point in the plane with coordinates x and y as (x, y), and similarly for points or vectors in higher-dimensional spaces.

Defining sets. More complicated and interesting sets are usually created from known sets using some rule. The sets of all squares of natural numbers can be written

$$\{i^2\colon i \in \mathbf{N}\}$$

or also

$$\{n \in \mathbf{N}\colon \text{ there exists } k \in \mathbf{N} \text{ such that } k^2 = n\}$$

or using the symbol $\exists$ for "there exists":

$$\{n \in \mathbf{N}\colon \exists k \in \mathbf{N}\,(k^2 = n)\}.$$

Another example is a formal definition of the open interval (a, b) introduced earlier:

$$(a, b) = \{x \in \mathbf{R}\colon a < x < b\}.$$

Note that the symbol (a, b) may mean either the open interval, or also the ordered pair consisting of a and b. These two meanings must (and usually can) be distinguished by the context. This is not at all uncommon in mathematics: many symbols, like parentheses in this case, are used in several different ways. For instance, (a, b) also frequently denotes the greatest common divisor of natural numbers a and b (but we avoid this meaning in this book).

With modern typesetting systems, it is no problem to use any kind of alphabets and symbols including hieroglyphs, so one might think of changing the notation in such cases. But mathematics tends to be rather conservative and the existing literature is vast, and so such notational inventions are usually short-lived.

The empty set. An important set is the one containing no element at all. There is just one such set, and it is customarily denoted by $\emptyset$ and called the *empty set*. Let us remark that the empty set can be an element of another set. For example, $\{\emptyset\}$ is the set containing the empty set as an element, and so it is not the same set as $\emptyset$!

Set systems. In mathematics, we often deal with sets whose elements are other sets. For instance, we can define the set

$$M = \{\{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{4\}\},$$

whose elements are 4 sets of natural numbers, more exactly 4 subsets of the set $\{1, 2, 3, 4\}$. One meets such sets in discrete mathematics quite frequently. To avoid saying a "set of sets", we use the notions *set system* or *family of sets*. We could thus say that M is a system of sets on the set $\{1, 2, 3, 4\}$. Such set systems are sometimes denoted by calligraphic capital letters, such as $\mathcal{M}$.

However, it is clear that such a distinction using various types of letters cannot always be quite consistent—what do we do if we encounter a set of sets of sets?

The system consisting of all possible subsets of some set X is denoted by the symbol[3] 2^X and called the *power set* of X. Another notation for the power set common in the literature is $\mathcal{P}(X)$.

Set size. A large part of this book is devoted to counting various kinds of objects. Hence a very important notation for us is that for the number of elements of a finite set X. We write it using the same symbol as for the absolute value of a number: $|X|$.

A more general notation for sums and products. Sometimes it is advantageous to use a more general way to write down a sum than using the pattern $\sum_{i=1}^{n} a_i$. For instance,

$$\sum_{i\in\{1,3,5,7\}} i^2$$

means the sum $1^2 + 3^2 + 5^2 + 7^2$. Under the summation sign, we first write the summation variable and then we write out the set of values over which the summation is to be performed. We have a lot of freedom in denoting this set of values. Sometimes it can in part be described by words, as in the following:

$$\sum_{\substack{i:\,1\le i\le 10\\ i\text{ a prime}}} i = 2 + 3 + 5 + 7.$$

Should the set of values for the summation be empty, we define the value of the sum as 0, no matter what appears after the summation sign. For example:

$$\sum_{i=1}^{0} (i + 10) = 0, \qquad \sum_{\substack{i\in\{2,4,6,8\}\\ i\text{ odd}}} i^4 = 0.$$

[3] This notation may look strange, but it is traditional and has its reasons. For instance, it helps to remember that an n-element set has 2^n subsets; see Proposition **??**.

A similar "set notation" can also be employed for products. An empty product, such as $\prod_{j:2\leq j<1} 2^j$, is always defined as 1 (*not* 0 as for an empty sum).

Operations with sets. Using the primitive notion of set membership, $\in$, we can define further relations among sets and operations with sets. For example, two sets X and Y are considered identical (equal) if they have the same elements. In this case we write $X = Y$.

Other relations among sets can be defined similarly. If X, Y are sets, $X \subseteq Y$ (in words: "X is a subset of Y") means that each element of X also belongs to Y.

The notation $X \subset Y$ sometimes denotes that X is a subset of Y but X is not equal to Y. This distinction between $\subseteq$ and $\subset$ is not quite unified in the literature, and some authors may use $\subset$ synonymously with our $\subseteq$.

The notations $X \cup Y$ (the union of X and Y) and $X \cap Y$ (the intersection of X and Y) can be defined as follows:

$$X \cup Y = \{z\colon z \in X \text{ or } z \in Y\}, \qquad X \cap Y = \{z\colon z \in X \text{ and } z \in Y\}.$$

If we want to express that the sets X and Y in the considered union are disjoint, we write the union as $X \dot{\cup} Y$. The expression $X \setminus Y$ is the *difference* of the sets X and Y, i.e. the set of all elements belonging to X but not to Y.

Enlarged symbols $\cup$ and $\cap$ may be used in the same way as the symbols $\sum$ and $\prod$. So, if $X_1, X_2, \ldots, X_n$ are sets, their union can be written

$$\bigcup_{i=1}^{n} X_i \tag{1.2}$$

and similarly for intersection.

Note that this notation is possible (or correct) only because the operations of union and intersection are *associative*; that is, we have

$$X \cap (Y \cap Z) = (X \cap Y) \cap Z$$

and

$$X \cup (Y \cup Z) = (X \cup Y) \cup Z$$

for any triple X, Y, Z of sets. As a consequence, the way of "parenthesizing" the union of any 3, and generally of any n, sets is immaterial, and the common value can be denoted as in (1.2). The operations $\cup$ and $\cap$ are also *commutative*, in other words they satisfy the relations

$$X \cap Y = Y \cap X,$$

$$X \cup Y = Y \cup X.$$

The commutativity and the associativity of the operations $\cup$ and $\cap$ are complemented by their *distributivity*. For any sets X, Y, Z we have

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z),$$

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

The validity of these relations can be checked by proving that any element belongs to the left-hand side if and only if it belongs to the right-hand side. The relations can be generalized for an arbitrary number of sets as well. For instance,

$$A \cap \left(\bigcup_{i=1}^{n} X_i \right) = \bigcup_{i=1}^{n} (A \cap X_i);$$

$$A \cup \left(\bigcap_{i=1}^{n} X_i \right) = \bigcap_{i=1}^{n} (A \cup X_i).$$

Such relations can be proved by induction; see Section **??** below. Other popular relations for sets are

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \quad \text{and} \quad X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

(the so-called *de Morgan laws*), and their generalizations

$$X \setminus \left(\bigcup_{i=1}^{n} A_i \right) = \bigcap_{i=1}^{n} (X \setminus A_i)$$

$$X \setminus \left(\bigcap_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (X \setminus A_i).$$

The last operation to be introduced here is the *Cartesian product*, denoted by $X \times Y$, of two sets X and Y. The Cartesian product of X and Y is the set of all ordered pairs of the form (x, y), where $x \in X$ and $y \in Y$. Written formally,

$$X \times Y = \{(x, y) \colon x \in X,\, y \in Y\}.$$

Note that generally $X \times Y$ is not the same as $Y \times X$, i.e. the operation is not commutative.

The name "Cartesian product" comes from a geometric interpretation. If, for instance, $X = Y = \mathbf{R}$, then $X \times Y$ can be interpreted as all points of the plane, since a point in the plane is uniquely described by an ordered pair of real numbers, namely its Cartesian coordinates[4]—the x-coordinate and the y-coordinate. This geometric view can also be useful for Cartesian products of sets whose elements are not numbers.

The Cartesian product of a set X with itself, i.e. $X \times X$, may also be denoted by X^2.

1.2 Graphs

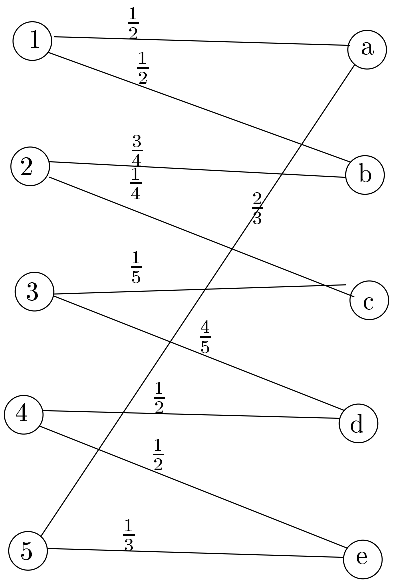
I assume that we are all familiar with the basic notions of Graphs, so I'll include the basic notation that we'll use.

- $V(G)$ denotes the vertex set of the graph G.
- $E(G)$ the edge set.
- $\overline{G}$ is the complement of G.
- $\alpha(G)$ will denote the independence number of G.
- $\chi(G)$ the chromatic number.
- $\Delta(G)$ is the maximal degree among the vertices of G.
- $\omega(G)(= \alpha(\overline{G}))$ is the clique number of G.
- $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ will denote the eigen values of G.

1.3 The Zero-Error capacity of noisy channels

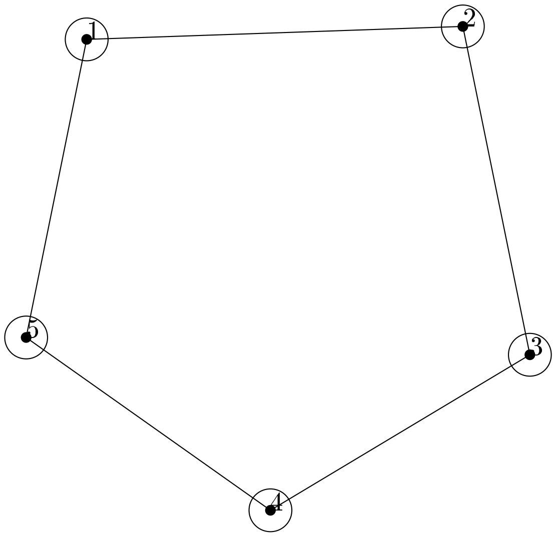
A noisy channel C is a triple $(X, Y; \psi)$ where X is a finite set and $\psi : X \Rightarrow 2^Y$ (the "noise"). In each atomic use of the channel, a sender sends a signal $x \in X$ and a receiver receives a signal $y \in Y$ selected arbitrarily from $\psi(x)$ so for instance, in the figure figure below as a result of sending the sequemce "1322435213" the sender may see the sequence "acccedacad" or the sequence "bdbbddacbc" and there is no way for him to know what did the sender send.

[4] These are named after their inventor, René Descartes.



If we wish to use the channel to communicate without error a sender and a receiver can agree on the following scheme: they select a subset $A \subset X$ such that $a, b \in A \Rightarrow \psi(a) \cap \psi(b) = \emptyset$. The sender sends only signals from A. No matter which signal the receiver sees, he'll know exactly the signal sent by the sender. So for instance, in the above example, the sender and reciever agree in advance that the sender will only use the signals “1” and “3”. So for instance, if the sender sends the sequence “1331113331” the reciever may still see different sequences, like “accbbaddcb” or “bdcabacdcb” but he'll always be able to determine precisely the sequence sent by the sender. Note that with this scheme, the reciever will never see the signal “e”.

This process can be naturally modeled as follows: we associate with a noisy channel $C = (X, Y; \psi)$ a graph $G(C)$, it's *characteristic graph*. $V(G) = X$ and two (distinct) vertices x_1, x_2 are connected by an edge if $\psi(x_1) \cap \psi(x_2 \neq \emptyset$.



Clearly, the largest number of error free signals that a channel C can communicate is $\alpha(G(C))$. The characteristic graphs of noisy channels have no particular structure. As a matter of fact, every graph is the characteristic graph of a noisy channel (see exercises 1 and 2).

Noisy channels can be combined in various ways to form new channels. For instance, given two channels C_1 and C_2 for each signal to be transmitted, the sender selects arbitrarily one of the two channels and sends a signal from this channel. The corresponding characteristic graph of the new channel will be the disjoint sum of the two graphs and the largest number of error free signals that this channel can communicate is $\alpha(G_1) + \alpha(G_2)$.

Another way to combine two noisy channels is as follows: we can form a new channel: $C = (X_1 \times X_2, Y_1 \times Y_2; \psi)$. In this new channel, a single atomic signal will be an ordered pair of signals. The corresponding characteristic graph is the *strong product* $G(C_1) \times G(C_2)$ formally defined as follows:

definition The strong product of two graphs G and H is the graph $G \times H$ where $V(G \times H) = V(G) \times V(H)$ and $(gh, g'h') \in E(G \times H)$ if $gg' \in= E(G)$ and $hh' \in= E(H)$ ($\in=$ means that either $g = g'$ or $gg' \in E(H)$).

Note that if $A \subset V(G_1)$ and $B \subset V(G_2)$ are independent sets then $A \times B$ is an independent set in $G_1 \times G_2$. This of course means that $\alpha(G_1 \times G_2) \geq \alpha(G_1)\alpha(G_2)$. Does equality always hold? Definitely not. The smallest example is the pentagon (C_5). We have $\alpha(C_5) = 2$ while $\alpha(C_5 \times C_5) = 5$.

It is easy to see that the strong products and sums are associative commutative and distributive. In particular this means that powers of a single channel are well defined. Hence we shall use the common notation G^n for The strong product of G with itself a times. We are now ready to define the basic notion of our Trip: the zero-error capacity of a noisy channel.

definition The information rate of a noisy channel C is $\log(\alpha(C))$.

defintion The zero error capacity of a graph G is $\theta(G) = sup(\alpha(G^n)^{1/n})$.

Exercises

1. Show that if a triangle free graph G is the characteristic graph of a noisy channel $C = (X, Y; \psi)$ then $|Y| \geq |E(G)|$.
2. Prove that every graph G is a characteristic graph of a noisy channel.
3. ** Prove that if G is connected then G^n is hamiltonian if n is "large enough".
4. *** Prove that if G is connected, $\Delta(G) \leq k$ then G^k is hamiltonian.
5. (a) Prove that $\alpha(C_7^2) = 10$
   (b) Prove that $\alpha(C_{4k+1}^2) = k(4k+1)$
   (c) Prove that $\alpha(C_{4k+3}^2) = 4k^2 + 5k + 1$
6. * Prove that if G is a graph with $\alpha(G) = 2$ then $\alpha(G^2) \leq 5$.
7. * Let $\mathbf{A(n) = \{(a_1, \ldots, a_{2n}) | a_i \in 0, 1, 2, 3, 4,\ a_{2i-1} < a_{2i}\ 1 \geq i \geq n\}}$. Let $\mathbf{B} \subset \mathbf{A(n)}$, $|\mathbf{B}| > \mathbf{4^n}$. Prove that there are two vectors b and c in $\mathbf{B}$ and a corresponding index k such that $\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\} = \emptyset$.
8. * Prove that $\alpha(C_9^3) = 81$
9. Let G be a graph with $V(G) = \binom{\{1,\ldots,7\}}{3}$.
   (a) Find $\alpha(G)$ and $\omega(G)$
   (b) Prove that G cannot be covered by $\alpha(G)$ cliques.
   (c) Prove that there is a collection of 10 cliques such that every vertex of G is included in exactly two cliques.