The Shannon Capacity of Graphs and related topics

Moshe Rosenfeld

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Lecture 2, Druhá Přednáška

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The strong product of graphs, basic properties

In this chapter we study basic combinatotrial properties of the Strong Product of graphs and explore related open problems.

We start by reviewing some basic observations made by Shannon in his 1956 paper and expose the open problems he mentioned in this paper. Note that since every graph is the characteristic graph of a noisy channel we can drop the noisy channels and talk about graphs only.

Recall that noisy channels (graphs) can be combined in two different ways:

1) Given n noisy channels $G_1, \ldots, G_n$ a word will be an n-tuple $x_1, \ldots, x_n$ where $x_i \in V(G_i)$
The characteristic graph of this new channel is defined as follows:
$V(G_1 \times G_2 \times \ldots \times G_n) = V(G_1) \times V(G_2) \times \ldots \times V(G_n)$.
$(g_1 \ldots g_n, g_1\prime \ldots g_n\prime) \in E(G_1 \times \ldots \times G_n$ iff $(g_i, g\prime_i) \in (=)G_i$ (recall that this is the strong product of graphs).

2) A word will be a single signal chosen arbitrarily from $G_1, \ldots, G_n$ The graph of this channel will be the disjoint union of the graphs $G_1, \ldots, G_n$.

1.0.1 Definition. $\theta(G) = \sup_n \sqrt[n]{\alpha(G^n)}$ *is called the* zero error capacity *of the graph* G.

From the information theory point of view we are facing a very simply stated question: for a given noisy channel what is the word length that will give us the best information rate for the given channel. For instance, we observed that for the pentagon words of length 2 yield a better rate than words of length 1. Immediately we can ask a few related questions:

- **Question 1.** For every (noisy channel) graph G is there a finite number $\psi(G)$ that yields the optimal word length?
- **Question 2.** Is $\theta(G) = \sqrt[k]{\theta(G^k)}$?
- **Question 3.** Is $\theta(G + H) = \theta(G) + \theta(H)$?

1.0.2 Theorem. $\alpha(G \times H) \geq \alpha(G) \times \alpha(H)$.

Proof Let $A \subset V(G)$ and $B \subset V(H)$ be independent sets of size $\alpha(G)$ and $\alpha(H)$ respectively. It is easy to see that $A \times B$ is an independent set in $G \times H$. $\diamondsuit$

Note that there are many examples for which strict inequality holds, such as the pentagon. We shall explore more examples later and in the exercises.

It is easy to see that $\alpha(K_n \times G) = \alpha(G)$. In his paper Shannon attempted to explore the question whether there are other graphs G for which $\alpha(G \times H) = \alpha(G)\alpha(H) \forall H$. We shall call such graphs *universal graphs*. Shannon gave a partial answer:

1.0.3 Theorem. *G is universal if $V(G)$ can be covered by $\alpha(G)$ cliques.*

Proof Let $\alpha(G) = k$ and let $C_1, \ldots C_k$ be k cliques such that $\cup_{i=1}^{k} C_i = V(G)$. Let A be an independent set in $G \times H$. Let $A_i \subset A = \{(g,h) | g \in A_i\}$. We first note that if $(g,h), (g',h') \in A_i$ then $h \neq h'$ and $(h,h') \notin E(H)$. In other words the set $H_i = \{h | (g,h) \in A_i$ is an independent set in H and also $|A_i| = |H_i| \leq \alpha(H)$. Hence $|A| = \sum_{i=1}^{k} |A_i| \leq k\alpha(H) \leq \alpha(G)\alpha(H)$. $\diamondsuit$

Remarks We first note that the smallest number of cliques that cover the vertices of the graph G is the chromatic number $\chi(\overline{G})$. Theorem [**?**] actually proves a bit more. It proves that $\alpha(G \times H) \leq \chi(\overline{G})\alpha(H)$ and $\theta(G) \leq \chi(\overline{G})$. In other words, $\theta(G)$ is bounded below and above by classical graph parameters whose computations complexity is NP-complete.

Also every regular bipartite graph G is universal and its zero-error capacity is $\frac{|G|}{2} = \alpha(G)$.

Shannon asked whether this condition is also necessary. We shall defer the answer to the next section. Shannon used this theorem to test and prove that all graphs of order upto 5 except for the pentagon are universal and therefore their zero-error capacity was easily determined. Shannon showed that $\sqrt{5} \leq \theta(C_5) \leq \frac{5}{2}$ and wondered what is the capacity of C_5. Besides the obvious question of computing the zero-error capacity in general, he also asked whether $\theta(G+H) = \theta(G) + \theta(H)$. Before we address these questions (all will be answered in the near future) we make some more observations.

1.0.4 Theorem. *(Ljubič)* $\theta(G) = \lim_{n\to\infty} \sqrt[n]{\alpha(G^n)}$.

Proof $\alpha(G \times H) \geq \alpha(G)\alpha(H) \Longrightarrow \alpha(G^{n+m}) \geq \alpha(G^n)\alpha(G^m)$.
Or: $\log\alpha(G^{n+m}) \geq \log\alpha(G^n) + \log\alpha(G^m)$. This means that the sequence $\{\log\alpha(G^n)\}$ is subadditive.
Since $\alpha(G^n) \leq |V(G)|^n$ $\frac{\log\alpha(G^n)}{n} \leq log|V(G)|$. By a famous theorem of Fekete $\lim\frac{\log\alpha(G^n)}{n} = \sup\log\frac{\log\alpha(G^n)}{n}$. ♢

1.1 Corollary. $\theta(G^k) = \theta^k(G)$.

Proof $\theta(G^k) = \lim\sqrt[n]{\alpha(G^{kn})} = \lim(\sqrt[nk]{(\alpha(G^{kn}))}^k = \theta^k(G)$. ♢

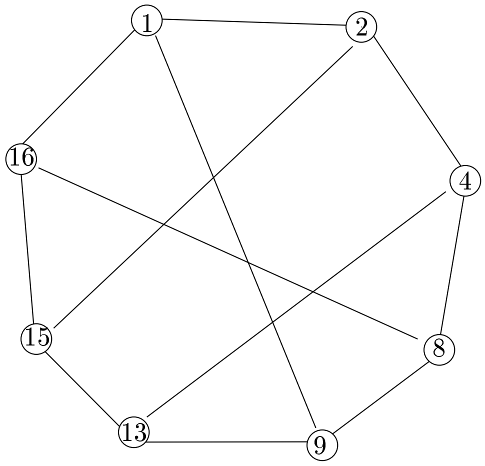
1.0.5 Theorem. $\alpha(G \times \overline{G} \geq |V(G)|$.

Proof The "diagonal" set $\{(g,g)|g \in V(G)\}$ is independent. ♢

The last theorem can be used to construct examples of graphs for which $\alpha(G \times H) > \alpha(G)\alpha(H)$. Indeed all we have to do is find graphs G, isomorphic to their complement $\overline{G}$ with $\alpha(G) < \sqrt{(|V(G)|}$ like the pentagon. A classical family of such graphs that provides a rich source of examples in many contexts in graph theory are the Paley graphs. Here is a description of a specific example:

The Paley graph $QR(17)$:
$V(G) = \{0, 1, \ldots, 16\}$
$E(G) = \{(n,m) \mid 0 \leq n, m \leq 16,\ n - m \in QR(17)\}$.

Some simple observations: $\phi_k : x \to k + x$ is an automorphism of this graph, hence it is vertex transitive.
$QR(17) = N(0) = \{1, 2, 4, 8, 9, 13, 15, 16\}$. This means that the subgraph spanned by $N(0)$ see below) is triangle free and $\omega(QR(17)) = 3$.



A simple calculation yields:
$\{3^k \pmod{17} \ \ 0 \leq k \leq 16\} =$
$\{1, 3, 9, 10, 13, 5, 15, 11, \ \ 16, 14, 8, 7, 4, 12, 2, 6\}$ (3 is called a *primitive root*. Every finite field has primitive roots).
But this means that $\psi : x \to 3x$ is an automorphism between $QR(17)$ and $\overline{QR(17)}$, in other words this graph is self-complemantary and therefore $\alpha(QR(17)) = 3$.

We can now ask how large can $\theta(G)$ be relative to $\alpha(G)$? At this point we are ready to tackle this question.

1.0.6 Theorem. *For every postive integer k, $\exists$ graphs G_k such that $\theta(G_k) \geq k\alpha(G_k)$.*

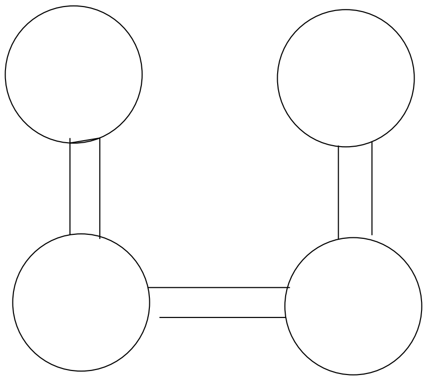
Proof Let G, H be graphs. The composition $G[H]$ is the graph defined by: $V(G[H]) = V(G) \times V(H)$
$E(G[H]) = \{(gh, g'h') | (g, g') \in E(G) \ or \ g = g' \ and \ (h, h') \in E(H)$.
It can be easily checked that:
1) The composition is associative.
2) $\alpha(G[H]) = \alpha(G)\alpha(H)$.
3) If G, H are self-complementary then so is $G[H]$.

Let G_0 be a self-complementary graph satisfying $\alpha^2(G_0) < |V(G)|$. Choose an inetger m such that $|V(G_0)|^m \geq k^2\alpha^{2m}(G_0)$ and let $G_k = G_0^{[m]}$. We have:
$\alpha(G_k) = \alpha^m(G_0)$. $\theta(G_k) = \sqrt{\theta(G_k^2)} \geq \sqrt{\alpha(G_k^2)} \geq \sqrt{|V(G_k)|} \geq k\alpha^m(G_0) = k\alpha(G_k)$. $\Diamond$

Remarks. The last theorem shows that $sup\frac{\theta(G)}{\alpha(G)} = \infty$. The rate of growth of this ratio is worth studying. If we use for G_0 the pentagon C_5 then we get self-complementary graphs G_k with 5^k vertices and $\alpha(G_k) = 2^k$.
V. Chvátál and P. Erdős were led by this construction to consider the question of estimating the largest size of a self-complementarty graph G with $\alpha(G) = k$. This graph cannot have more than $R(k, k)$ (the Ramsey number) vertices. They suggested the following:

1.0.7 Theorem. *There exist self-complementary graphs G with $|V(G)| = 4 \times 2^{\frac{k}{4}}$ and $\alpha(G) = k$.*

Proof It is known that $R(k, k) \geq 2^{\frac{k}{2}}$, i.e. there are graphs H such that $\alpha(H) \leq \alpha(\overline{H}) \leq \frac{k}{2}$ $|V(H)| \geq 2^{\frac{k}{4}}$. Let G be the graph shown in the figure below.



It is clear that $|V(G)| = 4|V(H)|$, $\alpha(G) = k$ and G is self-complementary. $\diamondsuit$.

This led V. Chátál and P. Erdős to conjecture that there is a self-complementary graph H with $\alpha(H) = k$ and $|V(H)| = R(k, k)$.

1.1 Hamiltonian cycles in G^n

This will be our first research project. Doufam, že tohle bude náš uspěšný společný projekt. In order to prepare for it we shall study a particular example: $K_{1,3}^2$ (see attached notes for its adjacency matrix). Note that the adjacency matrix of $K_{1,3}^2$ is $A \otimes A + A \otimes I + I \otimes A$. It is a simple technical matter to check that in general The adjacency matrix of $G \times H$ is $A_G \otimes A_H + A_G \otimes I + I \otimes A_H$.

Recall that G is $n-$connected if either $G = K_{n+1}$ or G cannot be separated by the removal of less than n vertices. It is $n-$connected between the distinct vertices g, g' If it contains $n-$vertex disjoint $g \leftrightarrow g'$ paths. Menger's celebrated Theorem states that G is $n-$connected iff it is $n-$connected between every pair of distinct vertices. The connectivity of G is the largest integer n such that G is $n-$connected. We denote this number by $\kappa(G)$. We have:

1.1.1 Theorem. $\kappa(G \times H) \geq min\{(\kappa(G)+1)(\kappa(H)+1)-1,\ |G|\kappa(H),\ \kappa(G)|H|\}$

Proof The proof is technical and nudni, so we'll spare ourselves the details. $\diamondsuit$

It follows that if G is connected then G^k is "highly" connected and its chances for being hamiltonian seem to be promising. Our first goal is to show that if G^{k_0} Is hamiltonian then so is G^k $\forall k \geq k_0$.

1.1.2 Theorem. *If G is hamiltonian and H a connected graph with $\Delta(H) \leq |G|$ then $G \times H$ is hamiltonian.*

Proof Without loss of generality we may assume that $G = C_n$ and $H = T$ is a tree. We proceed by induction on —T— to produce a particular hamiltonian cycle in $C_n \times T$.

Induction hypothesis: $C_n \times T$ contains a hamiltonian cycle that uses at least $n - d(t)$ edges of $C_n \times \{t\}$ $\forall t \in T$.

For $|T| = 1$ the induction claim clearly holds. Let T be a tree with $k+1$ vertices And let t_0 be a leaf of tree and t' its only neighbor in T. Let $T' = T \setminus \{t_0\}$. T' is a tree and $\Delta(T') \leq \Delta(T) \leq n$ hence by the induction hypothesis $C_a \times T'$ contains a hamiltonian cycle that uses at least $n - d_{T'}(t') \geq 1$ vertices of $C_n \times \{t'\}$. Let $((1,t_1),\ldots,(i,t'),(i+1,t'),\ldots,(x,y))$ be such a cycle. We modify it to: $C_n \times \{t'\}$. Let $((1,t_1),\ldots,(i,t'),(i,t_0),(i-1,t_0),\ldots(m,t_0),(i+1,t'),\ldots,(i+1,t_0),(i+1,t'),\ldots,(x,y))$. This cycle uses $n-1 = n - d_T(t_0)$ edges and $n - d_T(t')$ from $C_n \times T$. $\diamondsuit$

1.2 Corollary. *If G^k is hamiltonian then G^m is hamiltonian $\forall m \geq k$*

Proof $|G^k| \geq |G| > \Delta(G)$. $\diamondsuit$

For more observations we shall defer the discussion to the exercises. Thus for every connected graph G there might be a smallest integer $h(G)$ such that G^n is hamiltonian for all $n \geq h(G)$. The existence of $h(G)$ was proved by J.C.Bermond, Germa and Heydeman. Our goal is to prove that if G is connected then $G^{\Delta(G)}$ is hamiltonian.

The following questions may help us with our prject:

Exercises

1. Show that $K_{1,3}^2$ is not hamiltonian.

2. Show that : $K^3_{1,3}$ is hamiltonian
3. Show that for every positive integers k, n there is a k-connected graph G_k such that G^n_k is not hamiltonian.
4. Determine $h(K_{1,m})$.

1.2 Universal Graphs

We return now to the question of characterizing universal graphs. We shall obtain a complete characterization of universal graphs and show that Shannon's theorem [**?**] is sufficient but not necessary. We first need to recall some basic notions from Linear Programming.

Let $\mathbf{A}$ be an $n \times m$ matrix, $\mathbf{b}, \mathbf{c}$ two vectors of dimensions m, n respectively. The problem:

Maximize $< \mathbf{x}, \mathbf{c} >$ subject to $\mathbf{Ax} \leq \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}$ (the constraints) is called a **Linear Program**. Any vector $\mathbf{x}$ that satisfies the constraints is called a *feasible vector.*

Let G be a graph of order n and let $K_1, \ldots K_m$ be all the cliques of G. Let $\alpha_{i,j} = 1$ if $i \in K_j$ and 0 otherwise. This is the vertex-clique incidence function.

1.2.1 Definition. *Let* $\alpha^*(G) = max \sum_{i=1}^{n} x_i$ *over all vectors* $(x_1, \ldots x_n),\ x_i \geq 0,$ *and* $\sum_{i=1}^{n} \alpha_{i,j} x_i \leq 1,\ 1 \leq j \leq m.$

Note that $\alpha^*(G)$ is the objective function of a linear program with matrix $A = (\alpha_{i,j})$. We shall refer to this linear program as $P(G)$.

1.2.2 Theorem. *G is universal iff $\alpha^*(G) = \alpha(G)$.*

Proof Assume first that G is not universal and let H be a graph such that $\alpha(G \times H) > \alpha(G)\alpha(H)$. Let A be a maximal independent set in $G \times H$. For each vetex $i \in V(G)$ let $A_i = \{(i,h) | (i,h) \in A\}$ and $H_i = \{h | (i,h) \in A_i\}$. Note that since A_i is an independent set in $G \times H$, H_i is independent in H and $|H_i| = |A_i|$.
Let $x_i = \frac{|A_i|}{\alpha(H)}$. We first show that the vector $(x_1, \ldots, x_n)$ is feasible for the linear program $P(G)$. Let $C_t = \{i_1, \ldots i_t\}$ be a clique in G. Since C_t is a clique all vertices $h \in A_{i_s},\ 1 \leq s \leq t$ are distinct and also form an independent set in H. Hence $\sum_{i=1}^{n} \alpha_{i,t} x_i = \sum_{k=1}^{i_t} \frac{|A_{i_k}}{\alpha(H)} = \frac{1}{\alpha(H)} | \cup_{k=1}^{i_t} |H_{i_k}| \leq 1$. On the other hand, $\sum \alpha_{i,j} x_i \leq 1$ *and* $\sum_{i=1}^{n} x_i = \frac{\alpha(G \times H)}{\alpha(H)} > \alpha(G)$.

In the opposite direction assume that $\alpha^*(G) > \alpha(G)$. Since all the coefficients $\alpha_{i,j}$ of the linear program $P(G)$ are integers the optimal

vector $\mathbf{x} = (x_1, \ldots, x_n)$ may be chosen so that all it's coordinates are rational. Let $(x_1, \ldots, x_n) = (\frac{k_1}{m}, \ldots, \frac{k_n}{m})$ where m is the l.c.m of all the denominators of the coordinates and $k_i \geq 0$.
Define the graph H as follows:

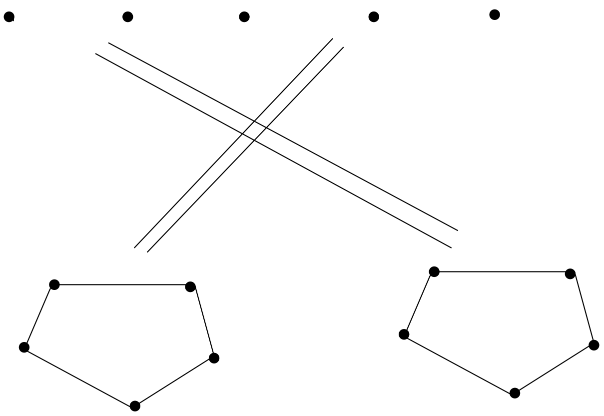
$V(H) = \cup H_i, \; |H_i| = k_i$
$(h, h') \in E(H) \; iff \; h \in H_s, h' \in H_t, (s,t) \notin E(G)$.

We need to show that $\alpha(G \times H) > \alpha(G)\alpha(H)$.

Let A be an independent set in H and let $K = \{k | H_k \cap A \neq \emptyset$. From the defintion of H it follows that K is a complete subgraph of G and thus contained in a clique K_j in G. We also have $|A| \leq |\cup_{k \in K} H_k|$. Since $\mathbf{x}$ is a feasible vector we have:
$\sum_{k \in K} |H_k| \leq \sum_{i=1}^{n} m\alpha_{i,j} x_i \leq m \Longrightarrow \alpha(H) \leq m$.

On the other hand it is easy to see that $B = \{(i,h) | h \in H_i\}$ is an independent set of vertices in $G \times H$ with $m\alpha^*(G)$ vertices. Thus $\alpha(G \times H) \geq m\alpha^*(G) > \alpha(G)\alpha(H)$. $\diamondsuit$

Once we have the full characterization of universal graphs we can show that Shannon's condition [**?**] is not necessary. Our first example is the graph G_1 shown below:



It is a simple matter to check that $\alpha(G_1) = 5$ and that the largest clique is a K_3. Since a pentagon can be covered by 3 but not by 2 cliques, this graph cannot be covered by 5 cliques. We also note that each vertex is contained in 10 cliques (triangles) and the graph contains 50 triangles. Hence:

$\sum_{j=1}^{50} \sum_{i=1}^{15} \alpha_{i,j} x_{i,j} = 10 \sum_{i=1}^{15} x_{i,j} \leq 50$. Or $\sum_{i=1}^{15} x_{i,j} \leq 5$.
Hence $\alpha^*(G_1) = 5$ so G_1 is universal and $\theta(G_1) = 5$.

The second example is more zajimavy. It is based on problem 9 from the first set of problems. Call this graph G_7. Two vertices are connected by an edge if the two triples have exactly one number in common. This graph has 35 vertices, $\alpha(G) = 5$ and $\omega(G) = 7$. A Fano plane or a Steiner Triple System on $\{1, \ldots, 7\}$ is a $7-$clique. Assuming the problem solved, denote the 10 cliques (each is a Fano plane) by $K_1, \ldots, K_{10}$. We have:

$\sum_{j=1}^{10} \sum_{i=1}^{35} \alpha_{i,j} x_{i,j} = 2 \sum_{i=1}^{35} x_{i,j} \leq 10 \Longrightarrow \alpha^*(G_7) = 5.$
So G_7 is universal and $\theta(G_7) = 5$.

With theorem [**?**] we extended our ability to calculate the Shannon capacity of graphs. Still, for all graphs for which we are able to find their capacity the optimal word length is 1. The smallest graph whose capacity cannot be determined by these methods is C_5.

Note that we have now an improved upper bound for $\theta(G)$. Unfortunately computing it is in NPC. Is there hope for computing $\theta(G)$ for more graphs or situace je beznadějna?

Before proving the next theorem we need a short review of some fundamental principles of Linear Programming.
The dual problem of the LP problem $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is:

Minimize $< \mathbf{y}, \mathbf{b} >$ subject to $\mathbf{y} \geq \mathbf{0},\ \mathbf{yA} \geq \mathbf{c}$.
The classical theorem states that if both problems have an optimal solution then both have the same value, that is $\min < \mathbf{y}, \mathbf{b} > = \max < \mathbf{x}, \mathbf{c} >$.

In computing $\alpha^*(G)$ we assign weights to vertices and try to maximize the total weight subject to the constraints that the total weight over every clique is ≤ 1. In the dual problem we wish to assign weights to all cliques and try to minimize the total weight subject to the constraint that for every vertex, the sum of the weights of the cliques containing it is ≥ 1. It is easy to see that every clique in $G \times H$ is the product of a clique from G by a clique from H and vice versa, every product of two such cliques is a clique in $G \times H$.
The tensor product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by:
$\mathbf{x} \otimes \mathbf{y} = (x_1 y_1, x_1 y_2, \ldots x_1 y_m, x_2 y_1, \ldots, x_n y_1, \ldots x_n y_m)$.
A simple calculation shows that:

$< \mathbf{a} \otimes \mathbf{b}, \mathbf{a}' \otimes \mathbf{b}' > = < \mathbf{a}, \mathbf{a}' >< \mathbf{b}, \mathbf{b}' >$.

1.2.3 Theorem. $\alpha^*(G \times H) = \alpha^*(G)\alpha^*(H)$.

Proof We first note that if $\mathbf{x}$ is a feasible vector for the graph G and $\mathbf{z}$ is feasible for H then $\mathbf{x} \otimes \mathbf{z}$ is feasible for $G \times H$. Since

$\sum_{i=1}^{a}\sum_{j=1}^{m} x_i z_j = (\sum_{i=1}^{a} x_i)(\sum_{j=1}^{m} z_j)$ we have:
$\alpha^*(G \times H) \geq \alpha^*(G)\alpha^*(H)$. By applying the same argument to the dual problem we get the reverse ineqaulity. $\diamondsuit$.

Comments. We have two programs that define the same parameter $\alpha^*(G)$. One maximizes a function (weights of vertices) and the other minimizes another function (weights of cliques). Thus to find $\alpha^*(G)$ all we need to find are two assignments of weights whose total is the same. The basic observations that made the last theorem work are the facts that any clique of the product $G \times H$ is a product of two cliques, any product of two cliques is a clique and the tensor product of a feasible vector for G by a feasible vector for H is feasible (this applies both to the primal program and the dual).

Exercises

1. Let $V(G) = GF(5^2)$ $E(G) = \{(x,y)|x - y = z^2\}$.
   a. Prove that $G \approx \overline{G}$
   b. Find $\alpha(G)$
2. Prove that if G is a graph with $\alpha(G) = 2$ then $\alpha(G^3) \leq 16$.
3. * Let $\binom{V(G)=\{1,\ldots,5\}}{5} \cup \{1,\ldots,5\} \cup \{0\}$
   $E(G) = \{(ab,cd)|\{a,b\} \cap \{c,d\} = \emptyset \cup (a,ab)|1 \leq a,b \leq 5 \ \cup (0,a)$.
   Prove that $\alpha(\overline{G}^3) = 16$
4. Consider a 7×7 square arrangement of 49 squares. Choose a square. Then choose a second, different square. You now created a 2-cell animal. This animal duplicates itself by a translation only so that the new animal occupies 4 squares. The process repeats until no more space is available. Is it possible to grow a 32-cell animal with this process? Justify your claim!
5. ** (library research) Give a complete proof that the Paley graph $QR(q)$, $q \cong 1 \pmod 4$ are self complementary.
   What is $\alpha(QR(q))$

Hints to selected exercises