The Shannon Capacity of Graphs and related topics

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Lecture 3, Třetí Přednáška

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The strong product of graphs and Ramsey Numbers

Ramsey's theorem states the following:

1.0.1 Theorem. *Given k positive integers $n_1, \ldots, n_k$, there is a finite number $R(n_1, \ldots, n_k)$ such that in every coloring of the edges of K_n, $n > R(n_1, \ldots, n_k)$, in k colors, there will be a color (index) i and a complete subgraph $K_m \subset K_n$ with $m \geq a_i + 1$ vertices such that all its edges will be colored i.*

The numbers $R(n_1, \ldots, n_k)$ are called *Ramsey Numbers*. Very few exact values of these numbers are known and the subject has been very popular in the last 50 years. We shall denote the Ramsey Numbers $R(m, m, \ldots, m)$ (m appears k times) by $R(m; k)$. The connection between Ramsey Numbers and the strong product was first observed by V. Chvátál and Z. Hedrlín and was later also observed by P. Erdős, R. McEliece and H. Taylor. In this section we shall explore this connection.

The following "naive" question will illustrate the depth of this connection.
Problem: *Does there exist a constant C such that every graph G with $\alpha(G) = 2$ $\theta(G) \leq C$?*

After all, we saw in the exercises that for such graphs $\alpha(G^2) \leq 5$ and $\alpha(G^3) \leq 16$. The answer to this question is actually equivalent to one of the classical open problems in Ramsey Theory: does there exist a constant C such that:

$$R(2; k) \leq C^k \ \forall k \geq 1?$$

From below, the current record is: $R(2; k) \geq c\sqrt[5]{321}^k$.

1.0.2 Theorem. *Let G_i, $1 \leq i \leq k$, be k graphs with $\alpha(G_i) = a_i$ then $\alpha(G_1 \times \ldots \times G_k) \leq R(a_1, a_2, \ldots, a_k)$.*

Proof Let A be an independent set in $G_1 \times \ldots \times G_k$ and let K_n be a complete graph with $V(K_n) = A$. We color the edges of K_n by k colors as follows: Since each pair of distinct vertices $(g_1, \ldots, g_k)$ and $(g_1^{'}, \ldots g_k^{'})$ are not connected by an edge, there is a first index i such that $g_i \neq g_i^{'}$ and $(g_i, g_i^{'}) \notin E(G_i)$. We color this edge by the color i. An i colored clique of size m in K_n will then correspond to an independent set of size m in G_i. Hence this is an edge coloring of K_n by k colors for which there is no i monochromatic complete subgraph of size $a_i + 1$ $i = 1, \ldots, k$ and therefore:

$$\alpha(G_1 \times \ldots \times G_k) = n \leq R(a_1, a_2, \ldots, a_k).$$

$\diamondsuit$

In the next two theorems we shall show that there is actually a very tight connection between the strong product and Ramsey Numbers.

1.0.3 Theorem. *Given k positive integers $a_1, \ldots, a_k$ there are graphs $G_1, \ldots, G_k$ with $\alpha(G_i) = a_i$ and $\alpha(G_1 \times \ldots \times G_k) = R(a_1, a_2, \ldots, a_k)$.*

Proof Let $R(a_1, a_2, \ldots, a_k) = m$ and let K_m be a complete graph on m vertices. By the assumptions, there is a coloring of the edges of K_m by k colors having no mono-chromatic complete i-edge colored subgraph of size $a_i + 1$.
Let $V(G_i) = V(K_m)$, $(i, j) \in E(G_i) \Leftrightarrow \chi(i, j) \neq i$. Obviously, $\alpha(G_i) \leq a_i$ (otherwise K_n would contain a monochromatic complete subgraph of size $\geq a_i + 1$).
On the other hand, the set $\{(i, i, \ldots, i) \mid i = 1, \ldots, m\}$ is an independent set in $(G_1 \times \ldots \times G_k)$ of size m. To see this we note that if $i \neq j$ then the edge $(i, j) \in E(K_m)$ is colored by some color s and therefore $(i, j) \notin E(G_s)$ and thus $((i, \ldots, i), (j, \ldots, j)) \notin E(G_1 \times \ldots, \times G_k)$.
Hence $\alpha(G_1 \times \ldots \times G_k) \geq R(a_1, a_2, \ldots, a_k)$ and by the previous theorem $\alpha(G_1 \times \ldots \times G_k) = R(a_1, a_2, \ldots, a_k)$. $\diamondsuit$

1.0.4 Theorem. *For every positive integers (m, k) there are graphs $G_{m,k}$ with $\alpha(G_{m,k}) \leq m$ and $\alpha(G^k_{m,k}) = R(m; k)$.*

Proof Let $G_1, \ldots, G_k$ be the graphs constructed in the previous theorem. Define the graph $G_{m,k}$ as follows:
$V(G_{m,k}) = \{(g, i) \mid g \in G_i, i = 1, \ldots k\}$;
$((g, i), (g', j)) \in E(G_{m,k})$ if $i \neq j$ or if $i = j$ and $(g, g') \in E(G_i)$.

Clearly, $\alpha(G_{m,k}) = m$. Since $G_{m,k}$ contains every G_i as an induced subgraph $G^k_{m,k}$ contains $G_1 \times \ldots \times G_k$ as am induced subgraph and thus $\alpha(G^k_{m,k}) \geq \alpha(G_1 \times G_2 \times \ldots \times G_k) = m$. By the previous theorem we must have $\alpha(G^k_{m,k}) = m = R(m;k)$. $\diamondsuit$

These observations lead to some interesting questions. The first is co bylo dřív e, slepice nebo vejce? Will we be able to discover new bounds on the Ramsey Numbers or will Ramsey Numbers will help us establish the capacity of certain graphs or both? Question number 3 in the second lecture notes is an example.

Conjecture Let G be the Clebsch graph (the complement of the graph described in question 3) then $\theta(\overline{G}) = \sqrt[3]{16}$.

This conjecture is interesting because it will be the first example of a graph for which the optimal word length is 3. Currently, there are no known examples of graphs for which the optimal word length is a finite number > 2.

If $\alpha(G) = m$ we have: $m^k \leq \alpha(G^k) \leq R(m;k)$. Another natural questions then is whether it is true that $\forall s$, $m^k \leq s \leq R(m;k)$ there is a graph G_s with $\alpha(G_s) = m$ and $\alpha(G^k_s) = s$?

We conclude this section with an observation that will prepare us for the surprising result that there are graphs G and H for which $\theta(G + H) >\neq \theta(G) + \theta(H)$.

1.0.5 Theorem. *(Noga Alon) Let $G = (V, E)$ be a graph of order m. Then $\theta(G + \overline{G}) \geq 2\sqrt{m}$.*

Proof Consider the following "large" set of vertices $A \subset (G + \overline{G})^{2n}$: each vertex v in this set is constructed as follows: we first choose arbitrarily n of the coordinates say $I_v = \{i_1 < i_2 \ldots < i_n\}$ and we place in them an arbitrary n-tuple of vertices from G. Let the remaining n coordinates be $I'_v = \{j_1 < j_2 < \ldots < j_n\}$. If $v_{i_k} = g \in V(G)$ then in the coresponding coordinate j_k we place $v_{j_k} = g \in \overline{G}$. Clearly, this set will contain $\binom{2n}{n}$ m^n vertiecs. If we select two distinct vertices u and v from this set then if $I_v \neq I_u$ there will be a coordinate s such that $v_s \in G$ and $u_s \in \overline{G}$ or vice versa. In either case $(u, v) \notin E(G + \overline{G})^{2n}$. If $I_v = I_u$ there must be an index r for which $v_{i_r} \neq u_{i_r}$. But then if $(v_{i_r}, u_{i_r}) \notin E(G)$ or if $(v_{i_r}, u_{i_r}) \in E(G)$ then $(v_{j_r}, u_{j_r}) \notin E(\overline{G})$ and in both cases $(u, v) \notin E(G + \overline{G})^{2n}$. Since A contains $\binom{2n}{n} m^n$ vertices, $\alpha(G + \overline{G})^{2n} \geq \binom{2n}{n} m^n$, but this implies that:

$$\theta(G + \overline{G}) = \lim_{n \to \infty} \sqrt[2n]{\binom{2n}{n} m^n} = 2\sqrt{m}.$$

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Thus Shannon's original question can be answered negatively if we can find graphs for which both the capacity of the graph and its complement are "small". But before we can produce them we need some more refined tools that will enable us to calculate $\theta(G)$ for more graphs than the tools developed so far.

Time permitting, we shall return to this subject later and discuss some of the beautiful constructions of graphs that lead to interesting Ramsey bounds and related capacity estimates.

Exercises

1. Prove that $\theta(G) + K_n = \theta(G) + 1$.
2. Prove that if $\chi(\overline{G}) = \alpha(G)$ then $\theta(G + H) = \theta(G) + \theta(H)$. (This was already observed by Shannon in 1956).
3. Can you prove the similar claim: if $\alpha^*(G) = \alpha(G)$ then $\theta(G + H) = \theta(G) + \theta(H)$.
4. * Prove that $\forall m,\ 9 \leq m \leq 17$, there is a graph G_m with $\alpha(G_m) = 3$ and $\alpha(G_m^2) = m$.
5. * Let $V(G) = \{(x_1, \ldots, x_8) \mid x_i = 0, 1\ \sum_{i=1}^{8} x_i = 2\}$. Connect two vertices u, v by an edge if $< u, v >= 0$. Find the eigenvalues of this graph.

Hints to selected exercises