The Shannon Capacity of Graphs and related topics

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Lecture 4, Čtvrtá Přednáška

1

Cycles, odd and even

This family, of some of the simplest graphs imaginable, is frustratingly challenging when it comes to the shannon capacity. Aside from the trivial odd cycle C_3 and even cycles which are universal graphs, only the capacity of the pentagon C_5 is known and it took 21 years of efforts of many researchers, before L. Lovász showed us how to do it by a brilliant idea whose ingenuity is in its simplictiy and power. We shall see this in the next set of lecture notes.
In this section we shall concentrate on the odd cycles and related packing problems in geometry and group theory. We shall try to establish some lower bounds for $\theta(C_{2k+1})$. Before we begin, we recall that the even cycles C_{2k} are universal graphs and thus $\theta(C_{2k}) = \alpha(C_{2k}) = k$. We also recall that if $\alpha(G^k) = m$ then $\theta(G) \geq \sqrt[k]{m}$, thus the numbers $\alpha(G^k)$, whenever they can be computed, give us lower bounds for $\theta(G)$.

1.0.1 Theorem. $\alpha(C^2_{4k+1}) = k(4k+1)$.

Proof Note first that $\alpha^*(C_{4k+1}) = \frac{4k+1}{2}$ and $\alpha(C_{4k+1}) = 2k$ hence by theorem xxxx $\alpha(C^2_{4k+1}) \leq k(4k+1)$. This means that all we have to do in order to prove the theorem is to produce an inedependent set in C^2_{4k+1} with $k(4k+1)$ vertices.
Let $A = \{(0,0),(0,2),\ldots,(0,2k-2),(1,2k),\ldots,(1,4k-2),$
(2,4k),(2,1), $\ldots$, (2, 2k -3), (3, 2k-1), (3,2k+1) , $\ldots$, $\ldots$, $(4k, 2k+1),\ldots,(4k,4k-1)\}$.

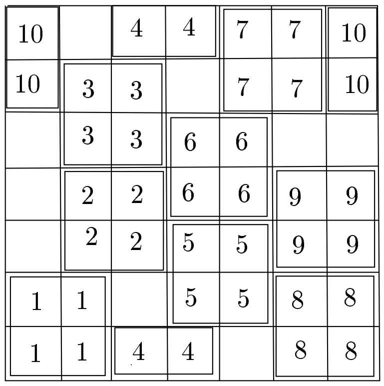
To understand the scheme, every vertex appears exactly k times in each coordinate. In the first coordinate, the vertex j is paired with k "consecutive" independent vertices $\{a, a+2,\ldots,a+2k-2\}$ then $j+1$ is paired with the "next" k "consecutive" independent vertices $\{a+2k,\ldots,a+4k-2\}$ etc. We start with 0 and pair it with the independent set $\{0,2,\ldots,2k-2\}$. Then for the next vertex 1 we shift (rotate) this set by adding 2 thus the combined vertices associated with 0 and 1 is the independent set $\{0, 2, \ldots, 4k\}$. For each subsequent vertex j we shift by 2 the k vertices associated with the vertex $j-1$ (all arithmetic is done $\pmod{4k+1}$). It is now a simple technical matter ro verify that A is independent and $|A| = k(4k+1)$. $\diamondsuit$

By a similar argument we can show that $\alpha(C_{2k+1}^2) = \lfloor \frac{k(2k+1)}{2} \rfloor$ and therefore $\theta(C_{2k+1}) \geq \sqrt{\lfloor \frac{k(2k+1)}{2} \rfloor} > \alpha(C_{2k+1})$. Since $\alpha^*(C_{2k+1}) = \frac{2k+1}{2}$, we have now the following bounds for the capacity of C_{2k+1}:

$$\tfrac{2k+1}{2} \geq \theta(C_{2k+1}) \geq \sqrt{\lfloor \tfrac{k(2k+1)}{2} \rfloor}.$$

Packing problems and the capacity of C_{2k+1}

Let T be a circle of length c. A *packing* of an open set $A \subset T$ is a family of disjoint subsets $\{A_1, \ldots, A_k\}$ each congruent to A. If A is an open arc of length γ then obviously one can pack exactly $\lfloor \frac{c}{\gamma} \rfloor$ such arcs in T. It is also clear that one can pack at least $\lfloor \frac{c}{\gamma} \rfloor^2$ squares in the torus T^2. The figure below shows a packing of 10 squares of size $\frac{2}{7} \times \frac{2}{7}$ in the torus T^2 thus showing that it may be possible to pack more than $\lfloor \frac{1}{\gamma} \rfloor^2$ in T^2.



If we denote by $c(\gamma, n)$ the number of $n-$cubes of side γ that can be packed in the $n-$dimensional torus T^n then we obviously have $c(\gamma, n+m) \geq c(\gamma, n)c(\gamma, m)$.

1.0.2 Theorem. $\alpha(C_{2k+1}^n) \leq c(\frac{2}{2k+1}, n)$.

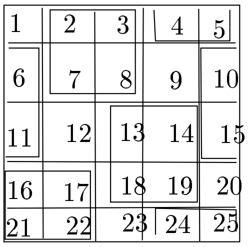
Proof Divide T into $2k+1$ equal arcs each of length $\frac{1}{2k+1}$ and assign the coordinates $\{0, 1, \ldots, 2k\}$ to the points of the subdivision. Let A be an independent set in C_{2k+1}^n. If $(x_1, \ldots, x_n) \in A$ construct an open $n-$cube with sides of length $\frac{2}{2k+1}$ centered at $(x_1, \ldots, x_n)$

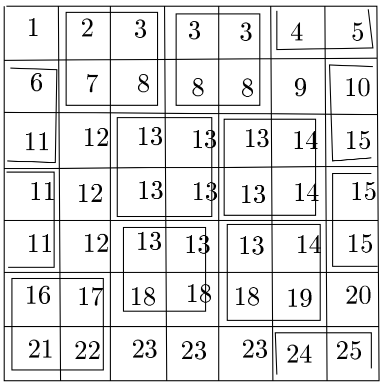
and faces "parallel" to the "axis". The independence of A assures that we get a packing of $|A|$ cubes in T^n. $\diamondsuit$

In the sequel, we shall restrict ourselves to packing of cubes centered at the lattice points of T^n, and aligned parallel to the "axis". Let $c_0(\frac{2}{k}, n)$ denote the maximal number of such cubes, centered at the lattice points that can be packed in T^n.

1.0.3 Theorem. $c_0(\frac{2}{2k+3}) \geq (\frac{2k+3}{2k+1})^n c_0(\frac{2}{2k+1}, n)$.

Proof We start with a fixed packing of $c_0(\frac{2}{2k+1}, n)$ n-cubes in T^n. We call each member of this set a *box*. The set $\{(x_1, x_2, \ldots, x_n) \in T^n,\ i_j \leq x_i \leq i_j + 1\}$ (a "unit" cube) will be called a *block*. Note that each *box* contains exactly 2^n *blocks*. A set of blocks that share exactly $2k$ coordinates is a *column*. Every *block* is contained in exactly n columns.
We label the blocks by the numbers $\{1, 2, \ldots, (2k+1)^n\}$. We start with a collection T_i^n, of $(2k+1)^n$ disjoint identically packed copies of T^n. Note that if a *block* labeled r is contained in a *box* then it will be contained in a *box* in all copies. In each copy T_i^n we perform the following "operation": each column containing *block* i will be repeated 2 additional time. All labels of blocks will be retained (see figure below). We duplicate the *boxes* originally contained in T_i^n as indicated in the figure. Thus whenever the *block* r was contained in a *box* it will also be contained in a *box* in the modified torus and also if i was contained in a *box* then in the modified T_i^n we include the additional box containing the "new" 2^n blocks labeled i.





The total number of *blocks* in the $(2k+1)^n$ copies is $(2k+1)^n(2k+3)^n$. Due to symmetry, each label appears the same number of times. Since we have $(2k+1)^n$ distinct original labels, each appears $(2k+3)^n$

times. It follows that the total number of labels which belong to *boxes* in all copies is $2^n c_0(\frac{2}{2k+1}, n)(2k+3)^n$. Since there are $(2k+1)^n$ disjoint copies of modified tori, at least one of them must contain $(\frac{2k+3}{2k+1})^n c_0(\frac{2}{2k+1}, n)$ *boxes.* $\diamondsuit$

Remark Clearly $c_0(\frac{2}{2k+1}, n) = \alpha(C^n_{2k+1})$.

1.1 Corollary. *If* $\theta(C_{2k_0+1}) = \frac{2k_0+1}{2}$ *then* $\theta(C_{2k+1}) = \frac{2k+1}{2}$ *for* $k \geq k_0$.

Proof $\alpha(C^n_{2k_0+3}) = c_0(\frac{2}{2k_0+3}, n) \geq (\frac{2k_0+3}{2k_0+1})^n c_0(\frac{2}{2k_0+1}, n)$
$\geq (\frac{2k_0+3}{2k_0+1})^n \alpha(C^n_{2k_0+1})$.

Thus $\lim_{n\to\infty} \sqrt[n]{\alpha(C^n_{2k_0+3})} \geq \frac{2k_0+3}{2k_0+1} \lim_{n\to\infty} \sqrt[n]{\alpha(C^n_{2k_0+1})} = \frac{2k_0+3}{2}$.

Since $\theta(C_{2k_0+3}) \leq \frac{2k_0+3}{2}$ we get that $\theta(C_{2k_0+3}) = \frac{2k_0+3}{2}$. By a simple induction the corollary follows. $\diamondsuit$

Unfortunately, the efforts to use this packing argument were thwarted by reality. As we shall see shortly, $\theta(C_{2k+1}) < \frac{2k+1}{2}$. But not all is lost, this theorem can be used to prove that $\lim_{k\to\infty}(\theta(C_{2k+1}) - \frac{2k+1}{2}) = 0$.

1.1 Free sets in Abelian groups

In this section we discuss yet another packing problem that will provide us with tools to calculate $\alpha(C^m_{2k+1})$ and as frequently happens, will also open a new set of interesting problems. To understand the "spirit" of these tools, we start with one of the previous exercises:

1.1.1 Lemma. $\alpha(C_9^3) = 81$.

Proof We first note that $\alpha(C_9) = 4$, $\alpha^*(C_9) = \frac{9}{2}$ and hence $\alpha(C_9^3) \leq 4\alpha^*(C_9)\alpha^*(C_9) = 81$. So all we need to do is produce an independent set of 81 vertices in C_9^3.

Let $N_9 = \{0, 1, \ldots, 8\}$. Define $\phi(x, y, z) = x + 2y + 4z \pmod 9$. Clearly, ϕ is a homomorphism between N_9^3 and N_9. Also, if $\{x, y, z\} \in \{0, \pm 1\}$ then $\phi(x, y, z) \neq (0, 0, 0)$ unless $(x, y, z) = (0, 0, 0)$.

Recall that $ker(\phi) = \{(x, y, z) \mid \phi(x, y, z) = 0\}$ is a subgroup of N_9^3 of order 81. But this subgroup does not contain any vector $(x, y, z) \neq (0, 0, 0)$ where $\{x, y, z\} \in \{0, \pm 1\}$. Viewed as a set of vertices in C_9^3 the subgroup forms an independent set of 81 vertices. $\diamondsuit$

1.1.2 Definition. *Let* $\mathbf{G}$ *be a finite abelian group. A subset* $F \subset \mathbf{G}$ *is called* free *if* $A, B \subset F$, $A \neq B$ *then* $\sum_{a\in A} a \neq \sum_{b\in B} b$.

For example, $A = \{1, 2, \ldots, 2^{k-1}\}$ is a *free* set in $\mathbf{N}_{2^k+1}$.

1.1.3 Definition. *$\tau(\mathbf{G}) = \{max|A| \mid A \subset \mathbf{G}\}$ is a free set.*

1.1.4 Theorem. $\tau(\mathbf{G}) \leq \lfloor \log_2 |\mathbf{G}| \rfloor$.

Proof If $A \subset \mathbf{G}$ is free the number of distinct subsets of A is $2^{|A|}$ and since distinct subsets sum to distinct elements of $\mathbf{G}$ we have: $2^{|A|} \geq |\mathbf{G}|$. $\diamondsuit$

1.1.5 Theorem. $\tau(\mathbf{G}) \geq \lceil \log_3 |\mathbf{G}| \rceil$.

Proof Assume that $A \subset \mathbf{G}$ is a maximal free set. Consider the expressions $\sum_{a_i \in A} \epsilon_i a_i$ where $\epsilon_i = 0, \pm 1$. If $b \neq \sum_{a_i \in A} \epsilon_i a_i$ then $A \cup \{b\}$ would be also free because if two distinct subsets A_1, A_2 would sum to the same element, we may assume that $A_1 \cap A_2 = \emptyset$. But then we would have: $\sum_{a_i \in A_1} a_i = \sum_{b_i \in A_2} b_i$. Now b must be included in exactly one of these subsets, assume in A_1 hence we would have $b = \sum_{a_i \in A} \epsilon_i a_i$ contradicitng our assumption. Hence if A is maximal every element of $\mathbf{G}$ can be expressed in the form $\sum_{a_i \in A} \epsilon_i a_i$ where $\epsilon_i = 0, \pm 1$. There are $3^{|A|}$ sums of this form thus $3^{|A|} \geq |\mathbf{G}|$ and $|A| \geq \lceil \log_3 |\mathbf{G}| \rceil$. $\diamondsuit$

Remark The last theorem leads to a backtracking approach when one attempts to write a computer program to evaluate $\tau(\mathbf{G})$.

Examples: $\mathbf{G} = N_3^n$ is actually the $n-$dimensional vector space over the field $GF(3)$. By the last theorem, A is free iff $\sum_{a_i \in A} \epsilon_i a_i = 0 \Longrightarrow \epsilon_i = 0$ but this is exactly linear independence over $GF(3)$ and hence $|A| \leq n$. Thus we have $\tau(\mathbf{G}) = n = \log_3 |\mathbf{G}|$.

If $\mathbf{G} = N_2^n$ a similar argument shows that $\tau(\mathbf{G}) = n = \log_2 |\mathbf{G}|$ thus both upper and lower bounds can be attained.

We denote by $\mathbf{G} \oplus \mathbf{H}$ the direct sum of the groups $\mathbf{G}$ and $\mathbf{H}$

1.1.6 Theorem. $\tau \mathbf{G} \oplus \mathbf{H} \geq \tau(\mathbf{G}) + \tau(\mathbf{H})$.

Proof If A and B are free sets in $\mathbf{G}$ and $\mathbf{H}$ then $\{(a, 0), (0, b) \mid a \in A, b \in B\}$ is a free set in $\mathbf{G} \oplus \mathbf{H}$. $\diamondsuit$

Clearly, $\tau(N_7) = 2$. The set $\{(1, 0), (0, 1), (1, 2), (1, 4), (3, 1)\}$ is a free set in $N_7 \oplus N_7$ showing that $\tau(\mathbf{G} \oplus \mathbf{H})$ can be strictly greater than $\tau(\mathbf{G}) + \tau(\mathbf{H})$.

1.1.7 Theorem. $\tau^*(\mathbf{G}) = \lim_{n \to \infty} \frac{\tau(\mathbf{G}^n)}{n}$.

Proof By the previous theorem, $\tau(\mathbf{G}^{n+m}) \geq \tau(\mathbf{G}^n) + \tau(\mathbf{G}^m)$ or the sequence $\{\tau(\mathbf{G}^n)\}$ is sub-additive. Since $\tau(\mathbf{G}^n) \leq \log_2 |\mathbf{G}^n| = nlog_2|\mathbf{G}|$ the sequence $\frac{\tau(\mathbf{G}^n)}{n}$ is bounded. By Fekete's theorem the limit exists and $\lim_{n \to \infty} \frac{\tau(\mathbf{G}^n)}{n} = \sup \frac{\tau(\mathbf{G}^n)}{n}$. $\diamondsuit$

Remark The last theorem opens a new can of worms, finding $\tau^*(\mathbf{G})$. We shall try some later.

We are now ready to connect these observations with the capacity of odd cycles. As is often the case, we shall observe some unexpected phenomena too. We first observe that if we view N_k^n as the vertices of the graph C_k^n then two vertices $(x_1, \ldots, x_n)$ $(y_1, \ldots, y_n)$ are connected by an edge iff $(x_1, \ldots, x_n) - (y_1, \ldots, y_n) = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{0, \pm 1\}$.
Also recall that if $\phi : \mathbf{G} \longrightarrow \mathbf{H}$ is a homomorphism from $\mathbf{G}$ onto $\mathbf{H}$ then $|\mathbf{G}| = |\mathbf{H}||ker(\phi)|$.

1.1.8 Theorem. *If $\tau(N_k^n) = m$ then $\alpha(C_k^n) \geq k^{m-n}$.*

Proof Before we prove this theorem, we should look at a couple of simple examples that will indicate its potential. As noted before, $\tau(N_{2^k+1}) = k$ and thus $\alpha(C_{2^k+1}^k) = (2^k + 1)^{k-1}$ (equality follows from $\alpha(C_{2^k+1}^*) = \frac{2^k+1}{2}$).

We also noted that $\tau(N_7^2) = 5$ and hence we get $\alpha(C_7^5) \geq 7^3$ or $\sqrt[5]{7^3} \leq \theta(C_7) \leq \frac{7}{2}$ this is an improvement of the bound previously obtained since $\sqrt{10} < \sqrt[5]{7^3}$.

Both lower bounds are improvements over the lower bounds given by $\alpha(C_k^2)$.

We are ready to start the proof. Let $\{f_1, f_2, \ldots, f_m\}$ be a maximal free set in N_k^n. Define a homomorphism $\phi : N_k^m \longrightarrow N_k^n$ by $\phi(x_1, \ldots, x_m) = \sum_{i=1}^m x_i f_i$. Clearly, ϕ is a homomorphism. Recall that $ker(\phi) = \{\mathbf{x} \in N_k^m \mid \phi(\mathbf{x}) = 0\}$ is a subgroup of order k^{m-n} of N_k^m. We claim that viewed as a subset of C_k^m it is an independent set. Indeed if $\mathbf{x}, \mathbf{y} \in ker(\phi)$ then $\phi(\mathbf{x} - \mathbf{y}) = 0 \Longrightarrow \sum_{i=1}^m (x_i - y_i) f_i = 0$, but $\mathbf{x} \neq \mathbf{y}$, by theorem [**?**] at least one of the coefficients $x_i - y_i \notin \{0, \pm 1\}$. $\diamondsuit$

1.1.9 Theorem. $\tau(C_n^k) \leq \lfloor \log_2(n^{k-1}(n-1) \rfloor$ $n \equiv 1 \pmod 2$.

Proof Let $\tau(C_n^k) = m$ so $\alpha(C_n^m) \geq n^{m-k}$. Let us consider a packing of $c_0(\frac{2}{n}, m) = \alpha(C_n^m)$ *boxes* in T^m. Each column contains n *blocks*. Since n is odd each column must contain at least one *block* not included in a *box*. There are mn^{m-1} columns and since each *block* is contained in exactly m columns there must be at least n^{m-1} *blocks* that do not belong to a *box* and hence:

$$n^m - n^{m-1} \geq 2^m \alpha(C_n^m) \geq 2^m n^{m-k} \Longrightarrow m \leq \lfloor \log_2(n^{k-1}(n-1) \rfloor. \quad \diamondsuit$$

This theorem may give us a sharper bound than the $log_2|\mathbf{G}|$. Indeed considering C_5^4 we have $\lfloor \log_2 5^4 \rfloor = 9$ while $\lfloor \log_2 4 \times 5^3 \rfloor = 8$ hence $\tau(C_5^4) = 8$. Soon we shall be able to prove that $\tau(C_5^k) = 2k$.

In this paragraph we shall derive an old result in elementary number theory that is a useful tool in some combinatorial applications. Two samples of these applications will be exposed.

1.1.10 Lemma. *Let $q = p^r, p$ a prime. If $0 \leq n < q-1$ then $\sum_{x \in GF(q)} x^n = 0$.*

Proof For $n = 0$ we assume that $0^0 = 1$ and for this case we have $\sum_{x \in GF(q)} 1 = q = 0$.
If γ is a primitive element in $GF(q)$ then: $\sum_{x \in GF(q)} x^n = \sum_{x \in GF(q)} (\gamma x)^n = \gamma^n \sum_{x \in GF(q)} x^n$.
But $\gamma^n \neq 1$ so we must have $\sum_{x \in GF(q)} x^n = 0$. ♢

1.1.11 Lemma. *If $f(x_1, \ldots, x_n)$ is a polynomial of total degree $< n(q-1)$ then $\sum_{(x_1,\ldots,x_n) \in GF^n(q)} f(x_1, \ldots, x_n) = 0$.*

Proof Without loss of generality we may assume that $f(x_1, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ where $\sum_{i=1}^n \alpha_i < n(q-1)$. Hence at least one power $\alpha_i < q-1$.
Since $\sum_{(x_1,\ldots,x_n) \in GF^n(q)} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} = \prod_{i=1}^n (\sum_{x \in GF(q)} x^{\alpha_i})$ by the previous lemma the claim is proved. ♢

1.1.12 Lemma. *(Warnig 1935) If $f_i(x_1, \ldots, x_n)$ is a set of r polynomials in n variables over $GF(q)$, each of total degree t_i, and if $\sum_{i=1}^r t_i < n$ then the number N of common zeroes of the polynomials, satisfies $N \equiv 0 \pmod p$.*

Proof We first note that if all polynomials are homogenous then the system will have non trivial solutions.
Let $P(x_1, \ldots, x_n) = \prod_{i=1}^{2k+1} (1 - f_i^{q-1}(x_1, \ldots, x_n))$. Since for $a \neq 0 \in GF(q)$, $a^{q-1} = 1$ we have: $P(x_1, \ldots, x_n) = 1$ only when $f_i(x_1, \ldots, x_n) = 0$ $\forall i$ and otherwise $P(x_1, \ldots, x_n) = 0$ thus $\sum_{(x_1,\ldots,x_n) \in GF(q)} P(x_1, \ldots, x_n) = N$. But the total degree of $P(x_1, \ldots, x_n)$ is $\sum t_i < n(q-1)$ hence $N \equiv 0 \pmod p$ ♢

1.1.13 Theorem. *$\tau(N_5^k) = 2k$.*

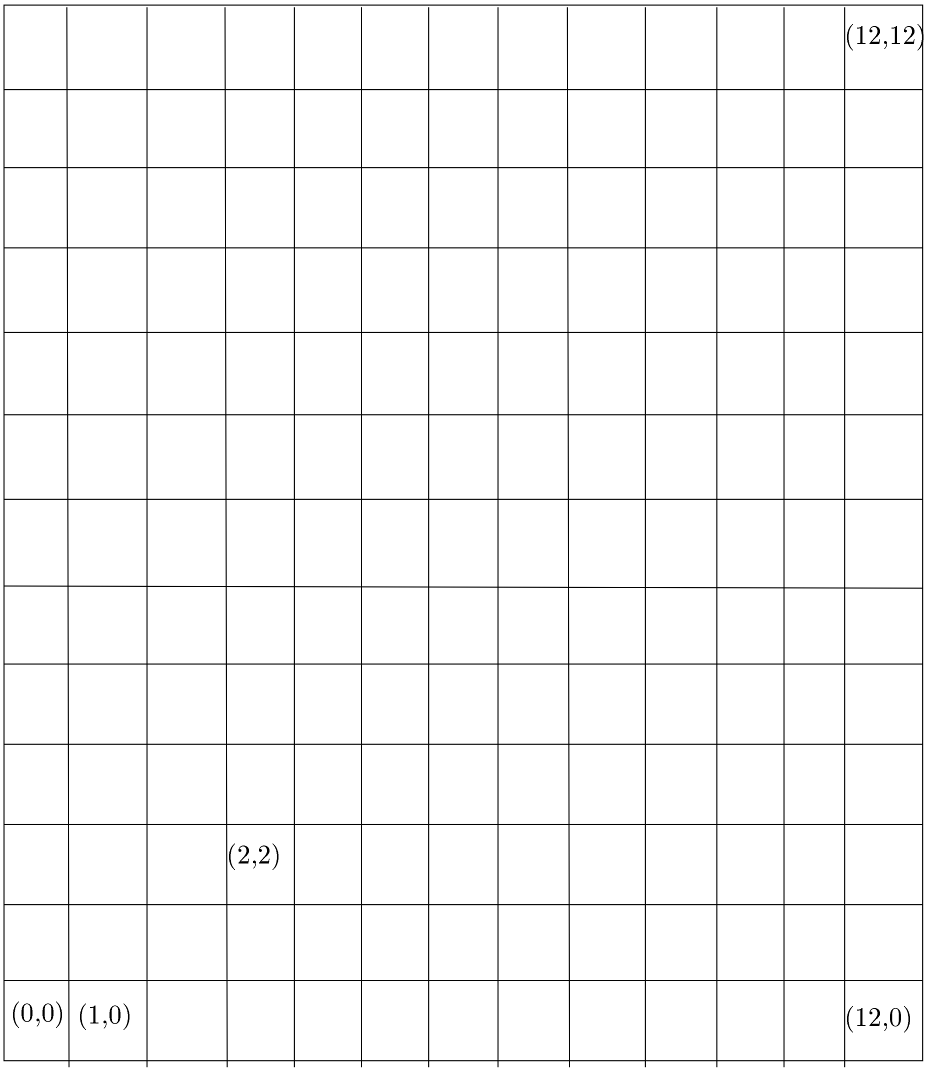
Proof Let $V_i = (a_{i,1}, \ldots, a_{i,k})$ $1 \leq i \leq 2k+1$ be $2k+1$ vectors in N_5^k. Let $f_j(x_1, \ldots, x_{2k+1}) = \sum_{i=1}^{2k+1} a_{i,j} x_i^2$ $j = 1, \ldots, k$. We have k polynomials in $2k+1$ variables. Each polynomial has total degree 2 hence by Waring's lemma the number of common zeroes is 0 (mod 5). Since $x_i = 0$ is a zero there must be non trivial zeroes. That is there are vectors $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$ that are common roots of all $2k+1$ polynomials. But if $0 \neq x \in GF(5)$ then $x^2 = \pm 1$ but this means that the $2k+1$ vectors cannot be free and thus $\tau(C_5^k) = 2k$. $\diamondsuit$

The next theorem, due to Alon, Friedland and Kalai, is yet another cute application of this method. I'll cite the result and leave it as an exercise.

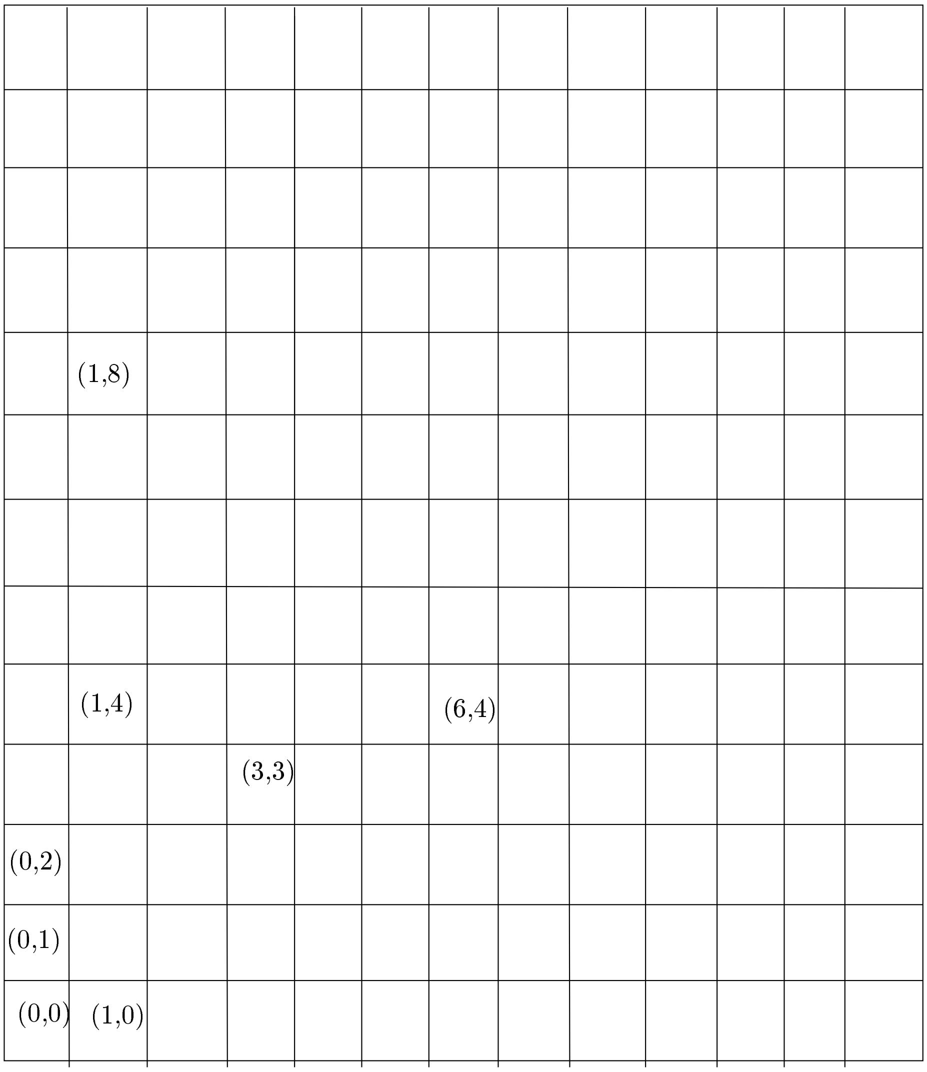
1.1.14 Theorem. *If G is a graph with $2|V(G)|+1$ edges then it contains a cubic subgraph.*

We conclude this section by translating the free sets to some entertaining puzzles that also lead to new results and problems. this will also be the basis of our next mini project. We shall concentrate on the plane but the results definitely generalize to higher dimensions.

Let $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a free set in C_n^2. Consider the torus T^2 with a grid coordinatized by $\{0, \ldots, n-1\}$ (see picture below).



If we start by covering the square $(0,0)$ then translate it by (x_1, y_1) to cover two squares, then translate this set by (x_2, y_2) to cover 4 squares and continue until we finish translating by all vectors in our free set, 2^k squares will be covered. This is guaranteed by the freedom of the initial set. We can then reverse the process, start with the geometric model (empty grid), and look for a way to "select" the translating squares. For example, the figure below shows a set of 7 free vectors in C_{13}^2.



Note that this implies that $\alpha(C_{13}^7) \geq 13^5$ and $\sqrt[7]{13^5} > \sqrt{39}$ hence an improved lower bound for $\theta(C_{13})$.

Mini project outline

Exercises

1. * Prove that if $3 \times 2^{n-2} \leq k < 2^n$ then $\tau(C_k^2) = \lfloor \log_2(k^2) \rfloor$.

2. * If G is a graph with $2|V(G)|+1$ edges then it contains a cubic subgraph.
3. Prove that if m, n are relatively prime than $\tau(C_m \times C_n) = \lfloor \log_2 mn \rfloor$.
4. How many sqaures can you "dynamically" cover in the tori $C_5 \times C_{14};\ \ C_5 \times C_{15}\ \ and\ C_6 \times C_{12}$?
5. *** Is $\tau(C_{47}^2) = 11$ or 10?
6. *** Is $\tau(C_7^5) = 13$?
7. ***** What is $\theta(C_7)$?

Exercises

1. Prove that $\alpha(C_{2k+1}^2) = \lfloor \frac{k(2k+1)}{2} \rfloor$
2. Prove that $\lim_{n\to\infty} \theta(C_{2^n+1}) - \alpha^*(C_{2^n+1}) = 0$.
3. Prove that $\beta(Z_{23}^2) = 8$.
4. Prove that $\beta(Z_7^3) = 7$.
5. ** Prove that the final dynamic packing of 32 squares in the 7×7 grid is unique upto rotations and reflections.

Hints to selected exercises