The Shannon Capacity of Graphs and related topics

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Lecture 5, Pátá Přednáška

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Eigenvalues, Strongly Regular Graphs and the Shannon capacity

In this chapter we shall review some basics of graph eigenvalues. We shall compute the eigenvalues of certain families of graphs, discuss strogngly regular graphs, review examples and end up by determining the shannon capacity of many more graphs. All these graphs will be regular and in most cases, the Shannon capacity will be determined by using the inequality $\vartheta(G) \leq \frac{n\lambda_n}{\lambda_n - \lambda_1}$ established in the previous chapter.

5.1 Graph eigenvalues

A complex number λ is an eigenvalue of the matrix $\mathbf{A}$ if there is a non-zero vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Every matrix has eigenvalues. With every square matrix $\mathbf{A} = (a_{i,j})$ we associate the quadratic form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i\overline{x_j}$.

5.1.1 Definition. *A symmetric matrix* $\mathbf{A}$ *is called* positive-semi-definte (PSD) *if* $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i\overline{x_j} \geq 0 \ \ \forall\mathbf{x}$.

Note that the correspondence between the quadratic form and the matrix A is not $1-1$. On the other hand, every quadratic form has a unique symmetric matrix associated with it. Thus when we speak about matrices associated with quadratic forms we shall always mean the unique symmetric matrix associated with it. From the definition we get that a diagonal matrix D in which $D_{ii} \geq 0$ is PSD and also if A and B are PSD so is $A+B$.

As we shall be interested primarily in symmetric matrices we start by recalling some basic facts from linear algebra: (we shall assume that $\mathbf{A}$ is symmetric)

- All eigenvalues of $\mathbf{A}$ are real.
- Eigenvectors belonging to distinct eigenvalues are orthogonal.
- The dimension of the eigenspace corresponding to an eigenvalue λ is equal to the multiplicity of λ as a root of the characteristic polynomial.
- $Trace(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.
- $Trace(\mathbf{A} \cdot \mathbf{A}^{tr}) \geq 0$
- $Trace(\mathbf{A} \cdot \mathbf{B}) = Trace(\mathbf{B} \cdot \mathbf{A})$
- $\mathbf{A}$ is PSD iff all eigenvalues of $\mathbf{A}$ are ≥ 0.
- $\mathbf{A} = \mathbf{TDT^{tr}}$ where $\mathbf{T}$ is an orthogonal matrix and $\mathbf{D}$ is a diagonal matrix.
- $\mathbf{A}$ is PSD then $\mathbf{A} = \mathbf{MM}^{tr}$ and $rank(\mathbf{A}) = rank(\mathbf{M})$.
- If λ_n is the smallest eigenvalue of A then $-\lambda_n \cdot I + A$ is PSD.
- If λ_1 is the largest eigenvalue of A then $\lambda_1 \cdot I - A$ is PSD.
- If $\mathbf{A}$ and $\mathbf{B}$ are PSD then $Trace(\mathbf{AB}) \geq 0$.
- If $\mathbf{A}$ has k distinct eigenvalues then there is a polynomial $p(x)$ of degree k such that $p(\mathbf{A}) = 0$.

We shall denote by $\mathbf{J}$ the matrix whose all entries are 1 and by $\mathbf{j}$ the vector $(1, 1, \ldots, 1)$. We begin by proving some facts about eigenvalues of adjacency matrices and PSD matrices.

5.1.2 Theorem. *Let G be a connected graph. If λ is an eigenvalue of G, then $|\lambda| \leq \Delta(G)$. Furtheremore, if $\lambda = \Delta(G)$ then it has multiplicity 1 and G is a regular graph.*

Proof Let $(x_1, \ldots, x_n)$ be an eigenvector corresponding to the eigenvalue λ. Choose x_i such that $|x_i| \geq |x_j|$. We may assume that $x_i = 1$. We have:
$|\lambda| = x_i|\lambda| = |\sum_{j=1}^{n} a_{i,j} x_j| \leq \sum_{j=1}^{n} a_{i,j} |x_j| \leq x_i deg(i) \leq x_i \Delta(G) = \Delta(G)$.

If $\lambda = \Delta(G)$ then the above inequalities imply that $deg(i) = \Delta(G)$ and for every vertex j adjacent to i, $x_j = 1$. Applying the same argument to the vertex j we also get that $deg(j) = \Delta(G)$. Since G is connected, we get that $deg(k) = \Delta(G)$ and the only corresponding eigenvector is the vector $\mathbf{j}$. $\diamondsuit$

5.1.3 Definition. *A strongly regular graph with parameters (k, a, b) is a graph G, regular of degree k in which every pair of adjacent*

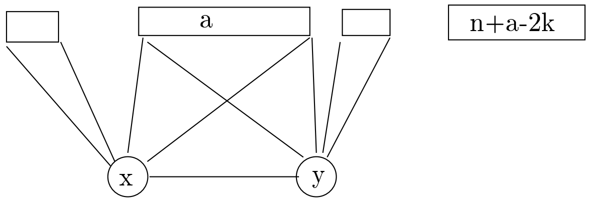
[0]The proof is simple: $\mathbf{A} = \mathbf{M} \cdot \mathbf{M^{tr}}$, $\mathbf{B} = \mathbf{P} \cdot \mathbf{P^{tr}}$ hence $Trace(\mathbf{AB}) = Trace(\mathbf{MM^{tr}PP^{tr}}) = Trace(\mathbf{M^{tr}PP^{tr}M}) = Trace((\mathbf{M^{tr}P})(\mathbf{M^{tr}P})^{\mathbf{tr}}) \geq 0$.

vertices have exactly a common neighbors and every pair of non-adjacent vertices have b common neighbors.

For example, C_5 is a $(2,0,1)$-strongly regular graph, the Petersen graph is a $(3,0,1)$-strongly regular graph and the Clebsch graph is a $(5,0,2)$-stongly regular graph. $L(K_n), L(K_{n,n})$ are also strongly regular graphs. In general, it is not known whether for a given triple of numbers (k,a,b) there is a strongly regular graph with these parameters. There are some non-trivial arithmetic conditions that govern the set of plausible triples but this is beyond the scope of our class. More examples will be discussed shortly.

5.1.4 Lemma. *If G is a (k,a,b)-strongly regular graph then $\overline{G}$ is a $(n-k-1, n-2k-2+b, n-2k+a)$ strongly regular graph.*

Proof Let $(x,y) \in E(G)$. Since G is k-regular, there are $k-a-1$ vertices that are connected by an edge to x and not connected by an edge to y. Hence there are $n-2-a-2(k-a-1) = n-2k+a$ vertices that are connected to neither x nor to y by an edge (see picture below).



But this means that in the complement, $(x,y) \notin E(\overline{G})$ have $n-2k+a$ common nieghbors. Similarly, if $(x,y) \notin E(G)$ there are $n-2-b-2(k-b) = n-2k-2+b$ vertices that are connected to neither x nor to y by an edge. In the complement we get that if $(x,y) \in \overline{G}$ then they have $n-2k-2+b$ $\diamondsuit$

5.1.5 Lemma. *If G is a (k,a,b)-strongly regular graph then $k(k-a-1) = (n-k-1)b$*

Proof We shall count the number of distinct copies of $K_{1,2}$ contained in G. We shall do this by two different ways. If $(x,y) \in E(G)$ then there are $k-a-1$ vertices that are connected by an edge to x but not to y. Hence the number of $K_{1,2}$'s in which x has degree 2 that include the vertex y is $k-a-1$. Since $deg(x) = k$, $nk(k-a-1)$ is twice the number of distinct copies of $K_{1,2}$ contained in G (each copy with root x is counted twice, once with the edge (x,y) and a second time with the edge (x,z)).

We now count the same number through the pairs of vertices not connected by an edge. Every such pair has b common neighbors. Each common neighbor determines exactly one copy of a $k_{1,2}$. G is k-regular and thus contains $\frac{n(n-k-1)}{2}$ pairs of distinct vertices that are not connected by an edge. Hence G contains $\frac{n(n-k-1)}{2}b$ distinct copies of $K_{1,2}$ or $k(k-a-1) = (n-k-1)b$ $\diamondsuit$

5.1.6 Theorem. *A non-complete (k, a, b)-strongly regular graph has three eigenvalues: k and $\frac{a-b\pm\sqrt{(a-b)^2+4(k-b)}}{2}$.*

Proof Let $\mathbf{A}$ be the adjacency matrix of G. It is easy to see that $\mathbf{A}^2 = a\mathbf{A} + b(\mathbf{J} - \mathbf{I} - \mathbf{A}) + k\mathbf{I} = (a-b)\mathbf{A} + (k-b)\mathbf{I} + b\mathbf{J}$. Since G is connected, k is a simple eigenvalue with correponding eigenvector $\mathbf{j}$. All other eigenvectors are orthogonal to it. Let $\mathbf{v}$ be an eigenvector corresponding to an eigenvalue $\lambda \neq k$. We have:
$\lambda^2\mathbf{v} = \mathbf{A}^2\mathbf{v} = (a-b)\mathbf{A}\mathbf{v} + (k-b)\mathbf{I}\mathbf{v} + b\mathbf{J}\mathbf{v} = (a-b)\lambda\mathbf{v} + (k-b)\mathbf{v}$ and since $\mathbf{v} \neq 0$ the eigenvalue λ must satisfy the quadratic:

$$\lambda^2 - (a-b)\lambda - (k-b) = 0.$$

The roots of this quadratic are the eigenvalues stated. $\diamondsuit$

5.1.7 Theorem. *If G is connected with diameter d then G has at least $d+1$ distinct eigenvalues.*

Proof If i, j are two vertices at distance k in G then $\mathbf{A}^k_{i,j} \neq 0$ while $\mathbf{A}^m_{i,j} = 0$ for $m < k$. From this it follows that the matrices $\mathbf{A}^0, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^d$ are linearly independent and therefore there is no polynomial $p(x)$ of degree $\leq d$ such that $p(\mathbf{A}) = 0$. On the other hand, if $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ are all eigenvalues of G then $p(\mathbf{A}) = 0$ for $p(x) = (x-\lambda_1)\ldots(x-\lambda_k)$, hence $k \geq d+1$. $\diamondsuit$

5.1.8 Theorem. *If G is a connected, k-regular graph with at most 3 distinct eigenvalues then G is strongly regular.*

Proof If G has only two distinct eigenvaules then its diameter is 1 and $G = K_n$. Assume that G has 3 eigenvalue. Then k is a simple eigenvalue with $c\mathbf{j}$ its only corresponding eigenvectors. Let $\alpha\ and \beta$ be the other eigenvalues. Let $p(x) = (x-\alpha)(x-\beta)$. Since the minimal polynomial of A is $(x-k)p(x)$ we have: $(A - k\mathbf{I})p(A) = 0$. But this means that every column of $P(A)$ is an eigenvector of $A - k\mathbf{I}$ or a multiple of $\mathbf{j}$. Since $p(A)$ is symmetric, so is every row. Hence $p(A) = A^2 + aA + b = c\mathbf{J}$ but this means that G is strongly regular. $\diamondsuit$

5.2 Calculating Eigenvalues

In preparation for establishing new bounds for the Shannon capacity of certain graphs and raising more interesting questions we shall compute the eigenvalues of some "classical" graphs. Each one of them has spme special significance which will be highlighted at the proper place.

- The Petersen Graph P.

  As noted above, Petersen's graph is a $(3, 0, 1)$-strogngly regular graph. By theorem [**?**] its eigenvalues are $\{3, 1, -2\}$. We also know that 3 is a simple eigenvalue and the sum of the eigenvalues is 0. Hence the eigenvalues of the Petersen graph are: $\sigma(P) = \{3^{\{1\}}, 1^{\{5\}}, (-2)^{\{4\}}\}$.
  Applying Lovász bound we get $\vartheta(G) \leq \frac{n\lambda_n}{\lambda_n - \lambda_1} = \frac{20}{5} = 4$ but $\alpha(P) = 4$ hence $\theta(P) = 4$.
  For the complement $\overline{P}$ we have: $\sigma(\overline{P}) = \{6^{\{1\}}, 1^{\{4\}}, (-2)^{\{5\}}\}$ and thus $\vartheta(\overline{P}) \leq \frac{n\lambda_n}{\lambda_n - \lambda_1} = \frac{5}{2}$. In this case we also have $\alpha^*(\overline{P}) = \frac{5}{2}$. Whether $\theta(\overline{P}) = \frac{5}{2}$ is an interesting open question.
- Clebsch graph Cl. The parameters are $(5, 0, 2)$ and again the eigenvalues are:
  $\sigma(Cl) = \{5^{\{1\}}, 1^{\{10\}}, (-3)^{\{5\}}\}$ so $\vartheta(Cl) = 6$. For the complement: $\sigma(\overline{Cl}) = \{10^{\{1\}}, 2^{\{5\}}, (-2)^{\{10\}}\}$. Thus $\vartheta(\overline{Cl}) = \frac{8}{3}$. For both graphs the capacity is not known, but I suspect that $\theta(\overline{Cl}) = \sqrt[3]{16}$.
- The Hoffman-Singleton graph SH. This graph, whose existence and construction will be discussed later, is a $(7, 0, 1)$-strongly regular graph of order 50. Its eigenvalues are $\{7, 2^{\{28\}}, (-3)^{\{21\}}\}$. The graph itself has $\alpha(HS) \geq 15$. But $\vartheta(HS) \leq \frac{50\cdot 3}{10} = 15$ hence $\alpha(HS) = \theta(HS) = 15$. This is a simple demonstration of using the Lovász bound to determine the independence number of a graph. More examples will be demonstrated later.
  Its complement $\overline{SH}$ has eigenvalues $\{42, 2^{\{21\}}, (-3)^{\{28\}}\}$ and $\alpha(\overline{HS}) = 2$. If indeed this is the right value then it will prove that the Ramsey number $R(k; 3) \geq (\frac{10}{3})^k$, which will be an improvement on the best known current bound.
- The Schläfli graph Sl. There are a few ways to define this graph. We shall adopt the most elementary combinatorial defnition that allows us a simple straight forward derivation of its combinatorial properties (this is not the way Schläfli found this graph).

Sl has 27 vertices labeled by 27 vectors $(x_1, \ldots, x_8) \in R^8$ partitioned into two sets A and B as follows:
For vertices in A, $x_i \in \{1, -3\}$, $x_1 = x_2 = 1$, $\sum_{i=1}^{8} x_i = 0$.
For vertices in B, $x_i \in \{-1, 3\}$, $x_1 + x_2 = 2$, $\sum_{i=1}^{8} x_i = 0$.
It is simple to verify that for any two distinct vertices the inner product of the vectors representing them is ± 8. Let $(u, v) \in E(Sl)$ if $<u, v>= 8$.
It is a simple technical matter to check that Sl is a $(16, 10, 8)$ strongly regular graph (left as an exercise). Hence by theorem [?] the eigenvalues of SL are $\{16, 4, -2\}$. Recall that 16 is a simple eigenvalue, $tr(Sl) = 0$ and we have that these eigenvalues have the multiplicities: $\{16, 4^{\{6\}}, (-2)^{\{20\}}\}$. Note that $rank(2I + Sl) = 7$. For the complement which is a $(10, 1, 5)$-strongly regular the eigenvalues are $\{16, 4^{\{6\}}, (-2)^{\{20\}}\}$. Thus for Sl we get $\vartheta(SL) \leq 3$ and since $\alpha(SL) = 3$ we have that $\theta(SL) = 3$. For the complement we have $\vartheta(\overline{SL}) \leq 9$. Soon, we shall be able to show that $\theta(SL) \leq 7$. As we shall see later, this graph is the "Petersen" of Shannon's capacity.

5.3 Paley Graphs

The Paley graphs form a rich source of graphs that exhibit many extremal properties. They provide a rich source of examples for many problems, in both directed and undirected graphs. In this section we explore some of their basic properties.

Let $GF(q)$ be the finite field on q elements where $q = p^r$, p prime. It is well known that the multiplicative group $GF^*(q)$ of $GF(q)$ is cyclic, that is there are elements $\alpha \in GF(q)$ such that $GF^*(q) = \{\alpha^0, \alpha, \ldots, \alpha^k, \ldots, \alpha^{q-2}\}$. These are called *primitive* elements, and since $GF^*(q)$ is cyclic there are quite a few of them. Let $QR(q) = \{\alpha^{2k} \mid 1 \leq k \leq \frac{q-1}{2}\}$. These are called the *quadratic residues*. The Quadratic Residue graphs, *Paley graphs*, are the graphs whose vertex set is $GF(q)$ and $(x, y) \in E$ if $x - y \in QR(q)$. Note that if $q \equiv 1 \pmod 4$ then $-1 \in QR(q)$ and thus if $x - y \in QR(q)$ so is $y - x \in QR(q)$. In this case the corresponding Paley Graph will be a graph

[0] Schläfli was a geometer in the 19-th century. This is not how he discovered this graph. His discovery was made in the 3-dimensional projective geometry over the complex numbers. He discovered that certain surfaces in this space, represented by "generic" homogeneous cubic polynomials $f(x, y, z, t) = \sum_{i_1+i_2+i_3+i_4=3} C_{i_1 i_2 i_3 i_4} x^{i_1} y^{i_2} z^{i_3} t^{i_4}$ contain 27 lines. His graphs consisted of these lines with edges between intersecting lines.

of order q regular of degree $\frac{q-1}{2}$. If $q \equiv 3 \pmod 4$ the Paley Graphs will be tournaments (complete, directed graphs).

In this paragraph we shall establish the "profiles" of the Paley graphs. Most of the information will be obtained through the automorphism group. Let $G = GF(q)$ be a Paley Graph ($q \equiv 1 \pmod 4$). Our first observation is that G is of order q, regular of degree $\frac{q-1}{2}$. With each $\beta \in GF^*(q)$ we associate the following two mappings:

- $\psi_\beta : x \longrightarrow x + \beta$.
- $\phi_\beta : x \longrightarrow x\beta$.

1. It is easily seen that ψ_β is an automorphism of G and therefore $Aut(G)$ acts transitively on the vertices. Thus to establish the independence number $\alpha(G)$ it will suffice to find the largest independent set that contains the vertex 0.
2. If $\beta \in QR(q)$ then $\phi_\beta \in Aut(G)$. To see this note that if $(x, y) \in E(G)$ then $x - y \in QR(q)$ and since $\beta \in QR(q)$ $\beta(x - y) \in QR(q)$ and thus $(\beta x, \beta y) \in E(G)$. The reverse is also true since $QR(q)$ is a subgroup of $GF^*(q)$. The automorphism ϕ_β leaves 0 fixed and maps $1 \longrightarrow \beta$. But this means that $Aut(G)$ acts transitively on the edges $(0, x)$. Let $(y, z) \in E(G)$ and let $\beta = z-y$. Then $(0, z-y) \in E(G)$ and $\psi_\beta : (0, y-z) \longrightarrow (y, z)$. In other words, $Aut(G)$ also acts transitively on the edges. This means that for every edge (x, y) the number of vertices connected to both x and y is independent on the edge (x, y). By a similar argument, ϕ_β also acts trasitively on pairs of vertices not connected by edge. Thus proving that G is a striongly-regular graph.
3. Let $\beta \notin QR(q)$. ϕ_β maps edges to non-edges and vice versa, hence it is an isomorphism between G and its complement $\overline{G}$.
4. We are now ready to calculate the parameters of $QR(q)$. Clearly, $k = \frac{q-1}{2}$. Since $QR(q)$ is self complementary from lemma 1.3.? we get that $n - 2k - 2 + b = a$ or $a - b = -1$ and from lemma 1.4.? we get that $a + b = k - 1$. Hence $QR(q)$ is a $(\frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$-strongly regular graph.
5. From theorem 1.4.? the eigenvalues of $QR(q)$ are $\{\frac{q-1}{2}, \frac{-1\pm\sqrt{q}}{2}\}$. Applying the Lovász bound we get $\vartheta(QR(q)) \leq \frac{q\frac{1+\sqrt{q}}{2}}{\frac{q-1}{2}+\frac{1+\sqrt{q}}{2}} = \sqrt{q}$. On the other hand, $QR(q)$ is self complementary and hence $\alpha(QR^2(q)) = \alpha(QR(q) \times \overline{QR(q)}) \geq q$ and hence:

$$\theta(QR(q)) = \sqrt{q} \quad \text{and} \quad \alpha(QR(q)) \leq \sqrt{q}$$

This is quite a "backdoor" way to prove that the largest set of elements in $GF(q)$ such that the diffrence between any two is a quadratic residue is $\leq \sqrt{q}$. It also provide us with examples of graphs of order n for which the largest independent set and largest clique are $\leq \sqrt{n}$.

5.4 Other useful computations of eigenvalues.

Let G be a graph. $G \oplus I_2$, the prism over G, is the cartesian product of G and I_2. Simply, it consists of two disjoint copies of G and "vertical" edges connecting "clones". Thus if G is an r-regular graph with n vertices $G \oplus I_2$ will be an $(r+1)$-regular graph with $2n$ vertices.

5.4.1 Theorem. *If $\sigma(G) = \{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of G then $\sigma(G \oplus I_2) = \{\lambda_1 \pm 1, \ldots, \lambda_n \pm 1\}$.*

Proof We first note that if A is the adjacency matrix of G then

$$B = \begin{pmatrix} A & I \\ I & A \end{pmatrix}$$

is the ajacency matrix of $G \oplus I_2$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the eigenvvectors of G and $\{\lambda_1, \ldots, \lambda_n\}$ the corresponding eigenvalues. it is now simple to verify that the $2n$ $2n-$dimensional vectors $\{(\mathbf{v}_i, \pm\mathbf{v}_i)\}$ are linearly independent and $(\mathbf{v}_i, \pm\mathbf{v}_i)B = (\lambda_i \pm 1)(\mathbf{v}_i, \pm\mathbf{v}_i)$. $\diamondsuit$

For a finale, we shall compute the eigenvalues of circulant graphs.

5.4.2 Definition. *G is called a circulant graph if every row of its adjacency matrix can be obtained by cyclically shifting the previous row by one place.*

For example the adjacency matrix of C_5 (see below) or for that matter any cycle C_n is a circulant graph.

$$A_{C_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

To compute the eigenvalues of a circulant graph of order n let $\omega_{k,n} = e^{\frac{2k\pi i}{n}}$, $k = 0, \ldots, n-1$ and $\mathbf{v}_k = (1, \omega_{k,n}, \ldots, \omega_{k,n}^j, \ldots, \omega_{k,n}^{n-1})$ The vectros $\{\mathbf{v}_i\}$ are linearly independent (their determinant is the Vandermonde). On the other hand, it is easy to see that $A\mathbf{v}_k =$

$(\sum_{i=2}^{n} a_{1,i}\omega_{n,k}^{i})\mathbf{v}_k$. For the cycles C_n we get:
$\sigma(C_n) = \{\omega_{k,n} + \omega_{n-k,n}\}$. In particular, $\lambda_n = 2\cos\frac{\pi}{n}$.

Exercises

1. * Prove that K_{97} can be decomposed into 4 isomorphic, vertex and edge transitive graphs. What is the maximum clique size of this graph?
2. * Let p be an odd prime number. Prove that $\theta(QR(p^2)) = \alpha(QR(p^2)) = p$.
3. Prove that $\vartheta(Cl) = 6$ where Cl is the Clebsch graph.
4. Let $V(G) = GF(16)$. Two vertices x, y are connected by an edge if $x + y = \beta^3$ for some $\beta \in GF(16)$. Finda $\alpha(G), \vartheta(G)$. Can you identify this graph?
5. Prove that the Schläfli graph Sl is a $(16, 10, 8)$-strongly regular graph.
6. Construct an orthogonal representation for Sl with stem vector $\mathbf{b}$ such that $\mathbf{b}^2 = 3$.