The Shannon Capacity of Graphs and related topics

Moshe Rosenfeld

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Lecture 7, Sedmá Přednáška

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Formulas for $\vartheta(G)$

In this chapter we shall obtain some alternative formulas for computing $\vartheta(G)$, use them to compute the capacity of graphs and see how they can be applied. In particular, we shall be able to indentify families of regular graphs for which $\vartheta(G) = \frac{n\lambda_n}{\lambda_n - \lambda_1}$.

7.0.1 Theorem. $\vartheta(G) = \min \lambda_{max}(A)$ *where the minimum is taken over all symmetric matrices A such that:*

$$A_{ii} = A_{ij} = 1 \textit{ when } (i,j) \notin E(G). \tag{7.1}$$

Proof Let A be a matrix satisfying the assumptions. Set $\lambda_{max} I - A = (\lambda_{max} - 1)I - A'$ ($A' = A - I$ is equal to A except that it has 0's along the diagonal). This matrix is PSD (clearly $\lambda_{max} - 1 > 0$), and therefore so is $I - \frac{1}{\lambda_{max}-1}A'$. Hence it produces a representation of G in S^d such that vertices not connected by an edge are at distance $a = \sqrt{2 + \frac{2}{\lambda_{max}-1}}$. By theorem 6.1.3 $\vartheta(G) \leq \frac{a^2}{a^2-2} = \lambda_{max}$. Note that for the stem vector $\mathbf{b}$, constructed from this representation, we have $<\mathbf{b}, \mathbf{u_i}>= 1$ for all vectors $\mathbf{u_i}$.

To prove that $\vartheta(G) \geq \lambda_{max}$ we need to construct a matrix A satisfying the assumptions for which $\vartheta(G) \geq \lambda_{max}(A)$.
Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthogonal representation with stem vector $\mathbf{b}$ such that $\mathbf{b}^2 = \vartheta(G)$. Let A be the $n \times n$ matrix defined by:
$A_{ii} = 1$
$A_{ij} = 1 - \frac{\mathbf{b}^2 <\mathbf{u}_i, \mathbf{u}_j>}{<\mathbf{b},\mathbf{u}_i><\mathbf{b},\mathbf{u}_j>}$ (note that $A_{ij} = 1 \; if \; (i,j) \notin E(G)$). Let $m_i = \frac{\mathbf{b}}{||\mathbf{b}||} - \frac{||\mathbf{b}||\mathbf{u}_i}{<\mathbf{b},\mathbf{u}_i>}$. The matrix $B = (<m_i, m_j>)$ is PSD. For $i \neq j$ we have:
$B_{ij} =<m_i, m_j>=<\frac{\mathbf{b}}{||\mathbf{b}||} - \frac{||\mathbf{b}||\mathbf{u}_i}{<\mathbf{b},\mathbf{u}_i>}, \frac{\mathbf{b}}{||\mathbf{b}||} - \frac{||\mathbf{b}||\mathbf{u}_j}{<\mathbf{b},\mathbf{u}_j>}>= 1 - \frac{||\mathbf{b}||<\mathbf{b},\mathbf{u}_i>}{||\mathbf{b}||<\mathbf{b},\mathbf{u}_i>} - \frac{||\mathbf{b}||<\mathbf{b},\mathbf{u}_j>}{||\mathbf{b}||<\mathbf{b},\mathbf{u}_j>} + \frac{\mathbf{b}^2<\mathbf{u}_i,\mathbf{u}_j>}{<\mathbf{b},\mathbf{u}_i><\mathbf{b},\mathbf{u}_j>} = -1 + \frac{\mathbf{b}^2<\mathbf{u}_i,\mathbf{u}_j>}{<\mathbf{b},\mathbf{u}_i><\mathbf{b},\mathbf{u}_j>} = -A_{ij}$.
The diagonal: $B_{ii} =<m_i, m_i>= 1 - 2 + \frac{\mathbf{b}^2}{<\mathbf{b},\mathbf{u}_i>^2} \leq \vartheta(G) - 1$. Hence

$\vartheta(G)\mathbf{I} - A = B + D$ where D is a diagonal matrix with non-negative entries and therefore $\vartheta(G)\mathbf{I} - A$ is PSD so $\vartheta(G) \geq \lambda_{max}(A)$. $\diamondsuit$

Comment: Since for $(i,j) \notin E(G)$ $A_{ij} = 1$ we get an embedding of G in S^d such that points corresponding to vertices not connected by an edge are at distance $\sqrt{2 + \frac{2}{\vartheta(G)}}$ apart. In other words, we have the following alternative equivalent geometric definiton of $\vartheta(G)$:

7.1 Corollary. *Let $\phi : V(G) \longrightarrow S^d$ be an embedding such that $(i,j) \notin E(G) \implies ||\phi(i) - \phi(j)|| = \beta$. Let $\gamma = \max\{\beta\}$ where the maximum is taken over all embeddings ϕ. Then $\vartheta(G) = \frac{\gamma}{\gamma-2}$.*

We can also use this theorem to identify a family of regular grpahs for which $\vartheta(G) = \frac{n\lambda_n}{\lambda_n - \lambda_1}$.

7.0.2 Theorem. *If G is a regular graph and its automorphism group Γ acts transitively on the edges of G then $\vartheta(G) = \frac{n\lambda_n}{\lambda_n - \lambda_1}$.*

Proof The matirx $A_x = \mathbf{J} - xA$ where A is the adjacency matrix of G satisfies the conditions 7.1 hence $\lambda_{max}(A_x) \geq \vartheta(G)$. The eigenvectors of G are $\mathbf{j}$ and $\mathbf{v}_2, \ldots \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, hence the eigenvalues of A_x are $n - x\lambda_1, -x\lambda_2, \ldots, -x\lambda_n$ and the largest is $\max\{n - x\lambda_1, -x\lambda_n\}$. If we choose x so that both are equal, that is $x = \frac{n}{\lambda_1 - \lambda_n}$, then clearly, $\lambda_{max}(A_x)$ will be minimized.

Let C be a matrix satisfying 7.1 such that $\lambda_{max}(C) = \vartheta(G)$ and let Γ be the automorphism group of G. Each automorphism in Γ can be represented by an $n \times n$ permutation matrix P. Let $C' = \frac{1}{|\Gamma|}\sum_{P\in\Gamma} PCP^{-1}$. Since Γ acts transitively on the edges $C'_{ij} = x$ $\forall(i,j) \in E(G)$. Hence $C' = \mathbf{J} - xA$. On the other hand, all matrices PCP^{-1} have the same spectrum and therefore $\lambda_{max}(C') \leq \lambda_{max}(C) = \vartheta(G)$ (this follows immediately from Rayleigh's principle $\lambda_{max}(A) = \max < A\mathbf{x}, \mathbf{x} >$ $||\mathbf{x}|| = 1$) but C' is of the form $\mathbf{J} - xA$ hence $\lambda_{max}(C') = \frac{n\lambda_n}{\lambda_n - \lambda_1} = \vartheta(G)$. $\diamondsuit$

Theorem 7.0.1 is an optimization process. The feasible region is definitely polyhedral but the objective function, $\lambda_{max}(A)$ is not linear. In the context of convex programming, this is a concave function and one can actually dualize this theorem. The next theorem is the convex dual of this theorem, but rather than spending time and efforts in developing the tools to dualize it we shall resort to the ingenious proof given by L. Lovász in his original paper.

7.0.3 Theorem. *Let G be a graph with $V(G) = \{1, \dots, n\}$. Let B range over all $n \times n$ PSD matrices such that:*

$$B_{ij} = 0 \; for \; (i,j) \in E(G) \tag{7.2}$$
$$tr(B) = 1 \tag{7.3}$$

Then $\vartheta(G) = \max_B tr(B\mathbf{J})$.

Proof Let A be a matrix satisfying the assumptions of theorem 7.0.1 such that $\lambda_{max}(A) = \vartheta(G)$. and let B be a matrix satisfyning the assumptions of this thorem. We have:
$tr(B\mathbf{J}) = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}A_{ij} = tr(AB)$ Since B and $\vartheta(G)I - A$ are PSD we have:
$\vartheta(G) - tr(B\mathbf{J}) = tr(\vartheta(G)B) - tr(AB) = tr((\vartheta(G)I - A)B) \geq 0$

In order to prove the reverse direction, we have to construct a matrix B satisfying the assumptions such that $tr(B\mathbf{J}) = \vartheta(G)$. Let $\{(i_1, j_1), \dots, (i_m, j_m)\}$ where $i_k < j_k$ be a fixed listing of all edges of G. Let $\psi : S^{n-1} \longrightarrow R^{m+1}$ be defined by:

$$\psi(\mathbf{h}) = \psi(h_1, \dots h_n) = (h_{i_1} \cdot h_{j_1}, \dots, h_{i_m} \cdot h_{j_m}, (\textstyle\sum_{i=1}^{n} h_i)^2).$$

Obviously ψ is a continous function and since S^{n-1} is closed and bounded so is $H = Conv(\psi(S^{n-1}))$.
Claim: $\zeta = (0, \dots, 0, \vartheta(G)) \in H$.
Assume this is not true, then since H is convex and compact it can be separated from ζ, that is there is a vector $\mathbf{a} = (a_1, \dots, a_m, y)$ and a constant α such that $< \mathbf{a}, \zeta > > \alpha$ while $< \mathbf{a}, \psi(\mathbf{h}) > \leq \alpha \; \forall \mathbf{h} \in S^{n-1}$. In particular, $< \mathbf{a}, \psi(1, 0, \dots, 0) >= y \leq \alpha$. On the other hand, $< \mathbf{a}, \zeta > > \alpha$ implies that $\vartheta(G)y > \alpha$ hence both y and α must be positive. By scaling, we may modify the vector $\mathbf{a}$ so that $y = 1$ and for the corresponding α we'll have $\alpha < \vartheta(G)$.
Define the matrix A by $A_{ii} = A_{ij} = 1$ for $(i,j) \notin E(G)$ and $A_{ij} = \frac{1}{2}a_k + 1$ when $(i,j) = (i_k, j_k)$ is an edge of G. We have: $< \mathbf{a}, \psi(\mathbf{h}) >= \sum_{k=1}^{m} a_k h_{i_k} h_{j_k} + (\sum_{i=1}^{n} h_i)^2 = \sum_{i=1}^{n} \sum_{i=1}^{n} A_{ij} h_i h_j \leq \alpha$.
Since the largest eigenvalue of A is $\max\{< A\mathbf{x}, \mathbf{x} >\} \; ||\mathbf{x}|| = 1\}$ we would have $\lambda_{max}(A) \leq \alpha < \vartheta(G)$ but the matrix A satisfies 7.1 and thus $\lambda_{max}(A) \geq \vartheta(G)$, a contradiction. Hence $(0, \dots, 0, \vartheta(G)) \in H$ as claimed.

This means that there are nonnegative numbers $\beta_1, \dots, \beta_N$ and vectors $\mathbf{h}_1, \dots, \mathbf{h}_N \in S_{n-1}$ such that:

$$\mathbf{h_k} = (h_{k_1}, \dots, h_{k_n}) \tag{7.4}$$

$$\sum_{i=1}^{N} \beta_i = 1 \tag{7.5}$$

$$\sum_{i=1}^{N} \beta_i \psi(\mathbf{h}_i) = \zeta \tag{7.6}$$

Define the matrix B by $B_{ij} = \sum_{k=1}^{N} \beta_k h_{k_i} h_{k_j}$. We have:
$tr(B) = \sum_{i=1}^{n} B_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{N} \beta_k h_{k_i} h_{k_i} = \sum_{k=1}^{N} \beta_k \sum_{i=1}^{n} h_{k_i} h_{k_i} = \sum_{k=1}^{N} \beta_k = 1$.
If $(i,j) \in E(G)$ then $\{i,j\} = \{i_s, j_s\}$ for some $s \leq m$. By looking at the s-th coordinate in (7.5) we get:

$$B_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{N} \beta_k h_{k_i} h_{k_j} = 0.$$

Finally, from the last coordinate in equation (7.5) we get: $\sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} = \sum_{i,j=1}^{n} \sum_{k=1}^{N} \beta_k h_{k_i} h_{k_j} = \sum_{k=1}^{N} \beta_k \sum_{i,j=1}^{n} h_{k_i} h_{k_j} = \sum_{k=1}^{N} (\sum_{i=1}^{n} h_{k_i})^2 = \vartheta(G)$

Remarks While it may not be easy to follow the steps in this proof in order to construct the optimal matrix B for a particular graph, we sometimes can "guess" it.
Example:. Consider the pentagon C_5. Let B be the matrix:

$$B = \begin{pmatrix} \frac{1}{5} & 0 & \frac{\sqrt{5}-1}{10} & \frac{\sqrt{5}-1}{10} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{\sqrt{5}-1}{10} & \frac{\sqrt{5}-1}{10} \\ \frac{\sqrt{5}-1}{10} & 0 & \frac{1}{5} & 0 & \frac{\sqrt{5}-1}{10} \\ \frac{\sqrt{5}-1}{10} & \frac{\sqrt{5}-1}{10} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{\sqrt{5}-1}{10} & \frac{\sqrt{5}-1}{10} & 0 & \frac{1}{5} \end{pmatrix}$$

We first note that $B_{ij} = 0$ when $(i,j) \in E(C_5)$, $tr(B) = 1$ and $tr(B\mathbf{J}) = \sqrt{5}$. In order to show that this is an optimal matrix we need to verify that B is PSD. Note that $B = \frac{1}{5}(\mathbf{I} + \frac{\sqrt{5}-1}{2}(\mathbf{J} - \mathbf{I} - A))$ where A is the adjacency matrix of C_5. Recall also that C_5 is a $(2,0,1)$-strongly regular graph and the eigenvalues of its adjacency matrix A are $\{2, \frac{-1\pm\sqrt{5}}{2}\}$.
$B\mathbf{j} = \frac{1}{5}(1 + 2\frac{\sqrt{5}-1}{2})\mathbf{j}$.
$B\mathbf{v}_{\frac{-1-\sqrt{5}}{2}} = \frac{1}{5}(1 + \frac{\sqrt{5}-1}{2}(-1 - \frac{-1-\sqrt{5}}{2}))\mathbf{v}_{\frac{-1-\sqrt{5}}{2}} = \frac{5-\sqrt{5}}{2}\mathbf{v}_{\frac{-1-\sqrt{5}}{2}}$
$B\mathbf{v}_{\frac{-1+\sqrt{5}}{2}} = \frac{1}{5}(1 + \frac{\sqrt{5}-1}{2})(-1 - \frac{-1+\sqrt{5}}{2})) = \mathbf{0}$
Hence B is PSD.

7.0.4 Lemma. *Let* $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ *and* $\{\overline{\mathbf{u}}_\mathbf{1}, \ldots, \overline{\mathbf{u}}_\mathbf{n}\}$ *be orthogonal representations for* G *and its complement* $\overline{G}$ *respectively. Then for any vectors* $\mathbf{c}, \mathbf{d}$ *we have:* $\sum_{i=1}^{n} < \mathbf{u_i}, \mathbf{c} >^2 < \overline{\mathbf{u}}_\mathbf{i}, \mathbf{d} >^2 \leq \mathbf{c^2 d^2}$.

Proof $< \mathbf{u_i} \otimes \overline{\mathbf{u}}_\mathbf{i}, \mathbf{u_j} \otimes \overline{\mathbf{u}}_\mathbf{j} > = < \mathbf{u_i}, \mathbf{u_j} > < \overline{\mathbf{u}}_\mathbf{i}, \overline{\mathbf{u}}_\mathbf{j} > = \delta_{i,j}$. In other words they form a system of n orthonormal vectors. Thus $\mathbf{c}^2\mathbf{d}^2 = (\mathbf{c} \otimes \mathbf{d})^2 \geq \sum_{i=1}^{n} (< \mathbf{c} \otimes \mathbf{d}, \mathbf{u}_i \otimes \overline{\mathbf{u}}_i >)^2 = \sum_{i=1}^{n} < \mathbf{u_i}, \mathbf{c} >^2 < \overline{\mathbf{u}}_\mathbf{i}, \mathbf{d} >^2$. $\diamondsuit$

7.2 Corollary. *If* $\{\overline{\mathbf{u}}_\mathbf{1}, \ldots, \overline{\mathbf{u}}_\mathbf{n}\}$ *is an orthogonal representation for* $\overline{G}$, *and* $\mathbf{d}$ *is a unit vector then* $\vartheta(G) \geq \sum_{i=1}^{n} < \overline{\mathbf{u}}_i, \mathbf{d} >^2$.

Proof In the previous lemma, select $\mathbf{c}$ to be an optimal stem vector for the orthogonal representation of G. $\diamondsuit$

7.3 Corollary. $\vartheta(G)\vartheta(\overline{G}) \geq n$.

Proof Apply lemma 7.0.4 to optimal stem vectros $\mathbf{c}$ for G and $\mathbf{d}$ for $\overline{G}$. $\diamondsuit$

The following min-max formula for $\vartheta(G)$ demonstrates a surprising close tie between G and $\overline{G}$.

7.0.5 Theorem. $\vartheta(G) = \max \sum_{i=1}^{n} < \mathbf{d}, \mathbf{v}_i >^2$ *where* $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ *range over all orthogonal representations of* $\overline{G}$ *and* $\mathbf{d}$ *over all unit vectors.*

Proof From corollary 7.2 $\vartheta(G) \geq \max \sum_{i=1}^{n} < \mathbf{d}, \mathbf{v}_i >^2$. In order to prove the theorem we need to construct an orthogonal representation $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ and a unit vector $\mathbf{d}$ for which the opposite inequality holds. Let $\mathbf{B}$ be a PSD matrix satisfying (7.2) and (7.3). Since $\mathbf{B}$ is PSD. $\exists\{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ such that $\mathbf{B}_{ij} = < \mathbf{w_i}, \mathbf{w_j} >$. If $\mathbf{w}_i = 0$ for some vertex i then by removing it from the graph G we obtain a subgraph G' with $\vartheta(G') = \vartheta(G)$ so the following argument would be applied to G' hence we may assume that $\mathbf{w}_i \neq 0$. We have:
$tr\mathbf{B} = \sum_{i=1}^{n} \mathbf{w}_i^2 = 1$ and $tr\mathbf{BJ} = \sum_{i,j} < \mathbf{w_i}, \mathbf{w_j} > = (\sum_{i=1}^{n} \mathbf{w}_i)^2$.
Let $\mathbf{u}_i = \mathbf{w}_i / ||\mathbf{w}_i||$ and $\mathbf{d} = \sum_{i=1}^{n} \mathbf{w}_i / || \sum_{i=1}^{n} \mathbf{w}_i ||$.
We get:

$$\sum_{i=1}^{n} < \mathbf{d}, \mathbf{u}_i > = (\sum_{i=1}^{n} \mathbf{w}_i^2) \sum_{i=1}^{n} < \mathbf{d}, \mathbf{u}_i > (\sum_{i=1}^{n} ||\mathbf{w}_i|| < \mathbf{d}, \mathbf{u}_i >)^2 =$$

$$(\sum_{i=1}^{n} < \mathbf{d}, \mathbf{w}_i >)^2 = (< \mathbf{d}, \sum_{i=1}^{n} \mathbf{w}_i >)^2 = (< \sum_{i=1}^{n} \frac{\mathbf{w}_i}{|| \sum_{i=1}^{n} \mathbf{w}_i ||}, \sum_{i=1}^{n} \mathbf{w}_i >)^2 =$$

$$(\sum_{i=1}^{n} \mathbf{w}_i)^2 = \vartheta(G).$$

(7.7)

In this derivation we used the Cauchy-Schwartz inequality: $(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \geq (\sum_{i=1}^{n} a_i b_i)^2$. $\diamondsuit$

7.0.6 Theorem. *Let* $\mathbf{A} = (a_{ij})$ *range over all symmetric matrices such that* $a_{ii} = 0\ \forall i$ *and* $a_{ij} = 0$ *when* $(i,j) \in E(G)$. *Let* $\{\lambda_1, \geq, \ldots, \geq \lambda_n\}$ *be the eigenvalues of* $\mathbf{A}$. *Then:*

$$\vartheta(G) = \max_{\mathbf{A}}\{1 - \frac{\lambda_1}{\lambda_n}\}.$$

Proof Note first that $\lambda_n < 0$ since $tr\mathbf{A} = 0$. Let $\mathbf{f} = (f_1, \ldots, f_n)$ be an eigenvector corresponding to λ_1 such that $\mathbf{f}^2 = -\frac{1}{\lambda_n}$. Consider the matrices $F = \ diagonal(f_1, \ldots, f_n)$ and $\mathbf{B} = F(\mathbf{A} - \lambda_n \mathbf{I})F$
$\mathbf{B}_{ij} = 0$ when $(i,j) \in E(G)$ and clearly $\mathbf{B}$ is PSD and also $tr\mathbf{B} - \lambda_n \sum_{i=1}^{n} f_i^2 = 1$. Hence by theorem 7.0.3 $\vartheta(G) \geq tr\mathbf{BJ} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i f_j - \lambda_n \sum_{i=1}^{n} f_i^2 = <\mathbf{Af}, \mathbf{f}> +1 = <\lambda_1 \mathbf{f}, \mathbf{f}> +1 = 1 - \frac{\lambda_1}{\lambda_n}$.
The reverse direction is left as an exercise. $\diamondsuit$
For example, for the pentagon C_5 we may use the adjacency matrix of its complement (which has the same eigenvlaues) and we obtain $\vartheta \leq 1 - \frac{(-1-\sqrt{5})/2}{2} = \sqrt{5}$.

7.0.7 Theorem. $\vartheta(G) \leq \alpha^*(G)$.

Proof Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orhtogonal representation of $\overline{G}$ and $\mathbf{c}$ a unit vector such that:
$\vartheta(G) = \sum_{i=1}^{n} <\mathbf{c}, \mathbf{u}_i>$ (theorem 7.0.5).
Let C be a clique in G. Then $\{\mathbf{u}_i \mid i \in C$ is an orthonormal set of vectors and hence $\sum_{i \in C} <\mathbf{c}, \mathbf{u}_i>^2 \leq \mathbf{c}^2 = 1$. Therefore the weights $<\mathbf{c}, \mathbf{u}_i>$ are feasible and $\alpha^*(G) \geq \sum_{i=1}^{n} <\mathbf{c}, \mathbf{u}_i> = \vartheta(G)$ $\diamondsuit$

7.4 Corollary. *(A. Hoffman) Let* $\lambda_1 \geq \ldots \geq \lambda_n$ *be the eigenvalues of the adjacency matrix of* G. *Then the chromatic number* $\chi(G)$ *is at least* $1 - \frac{\lambda_1}{\lambda_n}$.

Proof By theorem 7.0.7 applied to $\overline{G}$ we have: $\vartheta(\overline{G}) \geq 1 - \frac{\lambda_1}{\lambda_n}$. Since $\chi(G) \geq \alpha^*(\overline{G}) \geq \vartheta(\overline{G})$ the corollary follows. $\diamondsuit$

We conclude this chapter by showing that $\vartheta(G)$ is multiplicative and prove that $\vartheta(G)\vartheta(\overline{G}) = n$ when the automorphism group of G acts transitively on the edges.

7.0.8 Theorem. $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$.

Proof From the definiton of $\vartheta(G)$ and basic properties of stem vectors we have $\vartheta(G \times H) \leq \vartheta(G)\vartheta(H)$. To prove the reverse, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be orthogonal representations for $\overline{G}$ and $\overline{H}$ and $\mathbf{c}, \mathbf{d}$ unit vectors such that:

$\sum_{i=1}^{n} <\mathbf{c}, \mathbf{u}_i>^2 = \vartheta(G)$ and $\sum_{i=1}^{n} <\mathbf{d}, \mathbf{w}_i>^2 = \vartheta(H)$.

Then $\mathbf{u}_i \otimes \mathbf{w}_j$ is an orthogonal representation for $\overline{G} \times \overline{H}$ but $\overline{G} \times \overline{H} \subset \overline{G \times H}$ hence it is also an orthogonal representation for $\overline{G \times H}$. Since $\mathbf{c} \otimes \mathbf{d}$ is a unit vector we have:

$\vartheta(G \times H) \geq \sum_{i=1}^{n} \sum_{j=1}^{m} <\mathbf{u}_i \otimes \mathbf{w}_j, \mathbf{c} \otimes \mathbf{d}>^2$

$= \sum_{i=1}^{n} \sum_{j=1}^{m} (<\mathbf{u}_i, \mathbf{c}><\mathbf{w}_j, \mathbf{d}>^2 = \sum_{i=1}^{n} <\mathbf{u}_i, \mathbf{c}>^2 \sum_{j=1}^{m} <\mathbf{w}_j, \mathbf{d}>^2 = \vartheta(G)\vartheta(H)$. ♢

7.0.9 Theorem. *If G has a vertex transitive automorphism group Γ then:*

$$\vartheta(G)\vartheta(\overline{G}) = n \qquad\qquad \theta(G)\theta(\overline{G}) \leq n.$$

Proof Let $\mathbf{B}$ be a symmetric, PSD matrix satisfying (7.2) *and* (7.3) such that $tr\mathbf{BJ} = \vartheta(G)$ and let Γ be the automorphism group of G. Let $\mathbf{B}' = \frac{1}{|\Gamma|} \sum_{P \in \Gamma} P^{-1}BP$. Clearly, $tr\mathbf{B}' = 1$ and $tr\mathbf{B}'\mathbf{J} = \vartheta(G)$ it is also PSD, satisfies (7.2) *and* (7.3) and for all $P \in \Gamma$, $P^{-1}\mathbf{B}'P = \mathbf{B}'$. Since Γ acts transitively on the vertices, $\mathbf{B}'_{ii} = \frac{1}{n}$. Constructing the orthogonal representation for $\overline{G}$ as in the proof of theorem 7.0.9 we get: $\sum_{i=1}^{n} <\mathbf{d}, \mathbf{u}_i>^2 = \vartheta(G) = n <\mathbf{d}, \mathbf{u}_i>^2$. Hence $\sqrt{\frac{n}{\vartheta(G)}}\mathbf{d}$ is a stem vector for $\overline{G}$ and $\vartheta(\overline{G}) \leq \frac{n}{\vartheta}(G)$.

Since $\theta(G) \leq \vartheta(G)$ $\theta(G)\theta(\overline{G}) \leq n$. ♢

Exercises

1. Construct a symmetric, PSD matrix B that satisfies the conditions of theorem 7.0.3 for the Petersen graph with $tr(B\mathbf{J}) = 4$.
2. ** Let $K(n, r)$ be the Kneser graph $(n > 2r)$. Let $bfu \in R^n$ be a vector such that $\sum_{i=1}^{n} \mathbf{u}_i = 0$. For any r-subset $A \subset \{1, \ldots, n\}$ let $x_A = \sum_{i \in A} u_i$. Prove that the vector (x_A) is a characteristic vector of $K(n, r)$ and its corresponding eigenvalue is: -$\binom{n-r-1}{n-1}$.
3. * Prove that the multiplicity of the eigenvalue -$\binom{n-r-1}{n-1}$ is n.
4. ** Prove that the eigenvalue -$\binom{n-r-1}{n-1}$ is the smallest eigenvalue of $K(n, r)$ and hence $\binom{\alpha(K(n,r)) = \vartheta(K(n,r)) = n-1}{r-1}$.
5. Prove that for a graph G there is a matrix $\mathbf{A} = (a_{ij})$ such that $a_{ii} = 0$, $a_{ij} = 0$ when $(i, j) \in E(G)$ and such that $\vartheta(G) \leq 1 - \frac{\lambda_n(\mathbf{A})}{\lambda_1(\mathbf{A})}$.

6. * Let $V(G) = GF(2^4)$. $(x, y) \in E(G)$ if $x+y = z^3$. Prove that $\alpha(\overline{G}) = 2$ and $\theta(\overline{G}) \geq \sqrt[3]{16}$.

7. Let G be a graph with 19 vertices, regular of degree 6 such that $K_1 9 = 3G$. Prove that G contains a K_3 (triangle).