The Shannon Capacity of Graphs and related topics

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Orthogonal Representations of Graphs

We denote by S^d the unit sphere in R^{d+1} and by $<\mathbf{u}, \mathbf{v}>$ the inner product of two vectors in R^d.

6.0.1 Definition. *An orthogonal representation of a graph G is a map $\phi : V(G) \longrightarrow S^d$ such that $(x,y) \notin E(G) \Longrightarrow <\phi(x), \phi(y)> = 0$.*

We first note that every graph has many orthogonal representations. Indeed let A be an $n \times n$ matrix ($n = |V(G)|$), such that: $A_{i,i} = 1,\ A_{i,j} = 0$ if $(i,j) \notin E(G)$. Now choose the remaining entries small enough so that A will be a symmetric, positive-semi-definite matrix (this can be easily done). But this means that $A = MM^{tr}$ and the rows of the matrix M will form an orthogonal represenatation for G.

6.0.2 Definition. *Let (G, ϕ) be a graph with an orthogonal representation ϕ. $\mathbf{b}$ is a stem vector for (G, ϕ) if $< \phi(g), \mathbf{b} >^2 \geq 1\ \forall g \in V(G)$.*

Clearly, for each orthogonal representation, there is always a stem vector. Just select a vector that is not orthogonal to any vector $\phi(g)$ and make it dlouha.

6.0.3 Theorem. *Let (G, ϕ) be a graph and an orthogonal representation with stem vector $\mathbf{b}$. Then $\alpha(G) \leq \mathbf{b}^2$.*

Proof Let $\alpha(G) = k$ and assume that $\{1, 2, \ldots, k\}$ is an independent set. The vectors $\{\phi(1), \ldots, \phi(k)\}$ form an orthonormal set of vectors in R^d and thus can be extended to an orthonormal basis $\{\phi(1), \ldots, \phi(k), u_{k+1}, \ldots u_d\}$ Since $\mathbf{b} = \sum_{i=1}^{k} < \mathbf{b}, \phi(i) > \phi(i) + \sum_{i=k+1}^{d} < \mathbf{b}, u_i > u_i$ we get:
$\mathbf{b}^2 = \sum_{i=1}^{k} < \mathbf{b}, \phi(i) >^2 + \sum_{i=k+1}^{d} < \mathbf{b}, u_i >^2 \geq \sum_{i=1}^{k} < \mathbf{b}, \phi(i) >^2 \geq k$
$\diamondsuit$

This theorem hints at the advantages of alternative representations of graphs. In this case, independent sets of vertices (an abstract notion with no mathematical substance) are realized as orthonormal sets of vectors thus giving us access to the multitude of power tools from geometry and linear algebra. These ideas will be explored and used in the sequel.

6.0.4 Theorem. *Let G, H be two graphs with orthogonal representations (G, ϕ) and (H, ψ) respectively. Then $\phi(g) \otimes \psi(h)$ is an orthogonal representation for $G \times H$.*

Proof $||\phi \otimes \psi(g,h)||^2 = ||\phi(g) \otimes \psi(h)||^2 = <\phi(g) \otimes \psi(h), \phi(g) \otimes \psi(h)> = <\phi(g), \phi(g)><\psi(h), \psi(h)> = 1$.
If $(gh, g'h') \notin E(G \times H)$ then either $(g, g') \notin E(G)$ or $(h, h') \notin E(H)$ but this means that $<\phi \otimes \psi(g,h), \phi \otimes \psi(g', h')> = <\phi(g), \phi(g')><\psi(h), \psi(h')> = 0$ whenever $(gh, g'h') \notin E(G \times H)$. ♢

6.0.5 Theorem. *If (G, ϕ) is an orthogonal representation with stem vector $\mathbf{b}$, (H, ψ) an orthogonal representation with stem vector $\mathbf{d}$ then $(G \times H, \phi \otimes \psi)$ is an orthogonal representation with stem vector $\mathbf{b} \otimes \mathbf{d}$.*

Proof We saw in the previous theorem that $(G \times H, \phi \otimes \psi)$ is an orthogonal representation. We also have:
$<\mathbf{b} \otimes \mathbf{d}, \phi(g) \otimes \psi(h)>^2 = <\mathbf{b}, \phi(g)>^2<\mathbf{d}, \psi(h)>^2 \geq 1$. ♢

6.1 Corollary. *If (G, ϕ) is an orthogonal representation with stem vector $\mathbf{b}$ then $\theta(G) \leq \mathbf{b}^2$.*

Proof By the previous theorem $\mathbf{b}^{\otimes n}$ is a stem vector for G^n and by theorem [**?**] $(\mathbf{b}^{\otimes n})^2 \geq \alpha(G^n)$ but $(\mathbf{b}^{\otimes n})^2 = (\mathbf{b}^2)^n$ and we get:

$\theta(G) = \sup(\sqrt[n]{\alpha(G^n)}) \leq \sup(\sqrt[n]{(\mathbf{b}^{\otimes n})^2} = \mathbf{b}^2$. ♢

By developing tools for constructing orthogonal representations we shall be able to obtain many examples of graphs whose capacity is given by $\theta(G) = \sqrt{\alpha(G^2)} = \sqrt{|V(G)|}$.

The next theorem will give an alternative tool, more geometric in nature. It may appeal to those of us who prefer the geometric picture. It establishes yet another connection between a geometric representation of a graph and its capacity.

6.0.6 Theorem. *If $\phi : G \longrightarrow S^d$ satisfies $(x, y) \notin E(G) \Longrightarrow ||\phi(x) - \phi(y)||^2 = \alpha^2 > 2$ then G has an orhotgonal representation with stem vector $\mathbf{b}$ satisfying $||\mathbf{b}||^2 = \frac{\alpha^2}{\alpha^2 - 2}$.*

Proof $||u-v|| = \alpha \Rightarrow u^2 + v^2 - 2 <u,v>= \alpha^2$ or: $-2<u,v>= \alpha^2 - 2$.
We wish to "lift" ϕ to an orthogonal representation of G in S^{d+1} by the modification $\mathbf{u} \longrightarrow (\beta \mathbf{u}, \sqrt{1-\beta^2})$. We would like to choose β so that it will yield an orthogonal representation. From our assumptions, $(x,y) \notin E(G) \Leftrightarrow ||\phi(x) - \phi(y)||^2 = \alpha^2$. In the modified representation, $<(\beta\mathbf{u}, \sqrt{1-\beta^2}), (\beta\mathbf{v}, \sqrt{1-\beta^2})> = \beta^2 <\mathbf{u},\mathbf{v}> + (1-\beta^2)$. If we wish this to be an orthogonal representations, for pairs of vectors $\mathbf{u},\mathbf{v}$ at distance α we must choose β so that $-\beta^2(\frac{\alpha^2-2}{2}) + (1-\beta^2) = 0$ or: $\beta^2 = \frac{2}{\alpha^2}$.
hence $\sqrt{1-\beta^2} = \sqrt{\frac{\alpha^2-2}{\alpha^2}}$ and the vector $\mathbf{b} = (0,0,\ldots,0,\sqrt{\frac{\alpha^2}{\alpha^2-2}})$ is a stem vector satisfying $\mathbf{b}^2 = \frac{\alpha^2}{\alpha^2-2}$. ♢

This suggests new ways to try and find better upper bounds for $\theta(G)$. Indeed we are now ready to show the power of this beautiful idea introduced by L. Lovász in 1979 and prove that $\theta(C_5) = \sqrt{5}$.

An orthogonal representation of C_5 and its stem vector.

The five points $\{(\cos(\frac{2k\pi}{5}), \sin(\frac{2k\pi}{5})), 0 \le k \le 4\}$ on the unit circle in R^2 form the vertices of a regular pentagon. The distance d, between pairs of non adjacent points is $d^2 = 2 - 2\cos(\frac{4\pi}{5}) = 2 + 2\cos(\frac{\pi}{5})$.
From elementary trigonemetry, $\sin\alpha = \sin(\pi - \alpha)$ hence $\sin(\frac{\pi}{5}) = \sin(\pi - \frac{\pi}{5}) = \sin 4(\frac{\pi}{5})$. We use this equation to evaluate $\cos(\frac{\pi}{5})$:
$\sin(\frac{\pi}{5}) = \sin 4(\frac{\pi}{5}) = 2\sin 2(\frac{\pi}{5})\cos 2(\frac{\pi}{5}) = 4\sin(\frac{\pi}{5})\cos(\frac{\pi}{5})\cos 2(\frac{\pi}{5})$.
Or: $1 = 4\cos(\frac{\pi}{5})\cos 2(\frac{\pi}{5}) = 4\cos(\frac{\pi}{5})(2\cos^2(\frac{\pi}{5}) - 1)$ which yields the cubic equation:

$$8\cos^3\tfrac{\pi}{5} - 4\cos\tfrac{\pi}{5} - 1 = 0 \Longrightarrow \cos\tfrac{\pi}{5} = \tfrac{1}{\sqrt{5}-1}$$

(Verifying for the skeptics:
$8(\frac{1}{\sqrt{5}-1})^3 - 4(\frac{1}{\sqrt{5}-1}) = (\frac{1}{\sqrt{5}-1})(8(\frac{1}{\sqrt{5}-1})^2 - 4) = (\frac{1}{\sqrt{5}-1})(\frac{8}{6-2\sqrt{5}} - 4) = (\frac{1}{\sqrt{5}-1})(\frac{8\sqrt{5}-16}{6-2\sqrt{5}}) = 1$).
Hence $d^2 = 2 + \frac{2}{\sqrt{5}-1} = \frac{2\sqrt{5}}{\sqrt{5}-1}$. Thus C_5 has an orthogonal representation with stem vector $\mathbf{b}$ whose length satisfies $\mathbf{b}^2 = \frac{d^2}{d^2-2} = \sqrt{5}$ yielding:

$$\theta(C_5) = \sqrt{5}$$

This was the first graph discovered for which the optimal information rate is attained by words of length 2. Also, it was the first graph for which $\theta(G) \neq \alpha^*(G)$. Thus far, in this millenium, no graphs for which the optimal information rate is attained by words of length $k > 2$ are known, though we do have some "suspects". My favorite suspect is the complement of the Clebsch graph $\overline{CL}$. We proved that $\alpha(\overline{CL}^3) = 16$ and I believe that $\theta(\overline{CL}) = \sqrt[3]{16}$.

6.1 Orthogonal representations and eigenvalues

A very natural orthogonal representation for a graph G can be readily derived from its adjacency matrix A_G. Indeed if λ_n is the smallest eigenvalue of A_G then $aI + A_G = a(I + \frac{1}{a}A_G)$ where $a = -\lambda_n$ will be a PSD matrix. If we write $M \cdot M^{tr} = I + \frac{1}{a}A_G$ then the rows of M will form an orthogonal representation of G. Bohŭzel this representation does not give us a good useful information about stem vectors.
A slight variation though is much more useful. Let $\beta > 0$. Let λ be the smallest eigenvalue of the matrix $A(\alpha, \beta) = \alpha \cdot A_G + \beta(\mathbf{J} - A_G - I)$. Note that $trace(A(\alpha, \beta)) = 0 \Longrightarrow \lambda < 0$. The matrix $I + \frac{1}{\lambda}A(\alpha, \beta))$ is positive-semi-definite and it will give us an embedding of G in S^d such that $||i - j||^2 = 2 - \frac{2\beta}{\lambda} > 2$ when $(i, j) \notin E(G)$.
In particular, when $G = G(n, r)$ is an r-regular graph on n vertices, with eigenvalues $\{r = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$ we have an explicit expression for the eigenvalues of $A(\alpha, \beta)$):
$\sigma(A(\alpha, \beta)) = \{(\alpha \cdot r + \beta(n - r - 1), (\alpha\lambda_2 + \beta(-\lambda_2 - 1)), \ldots, (\alpha\lambda_n + \beta(-\lambda_n - 1))\}$

We can apply this to show that $\theta(P) = 4$ for the Petersen graph P. (Note that $\alpha(P) = 4$ and $\alpha^*(P) = 5$.) As we noted before, for the petersen graph we have: $\sigma(P) = \{3, 1, -2\}$. Let $\mathbf{v_3} = \mathbf{j}, \mathbf{v_1}, \mathbf{v_{-2}}$ be eigenvectors corresponding to these eigenvalues. So the eigenvalues of $P(-1, 1)$ are: $3, -3$ as the following simple calculation shows.

$(\mathbf{J} - \mathbf{I} - 2P)\mathbf{j} = 3\mathbf{j}$
$(\mathbf{J} - \mathbf{I} - 2P)\mathbf{v_1} = -3\mathbf{v_1}$
$(\mathbf{J} - \mathbf{I} - 2P)\mathbf{v_{-2}} = 3\mathbf{v_{-}2}$

Hence $I + \frac{1}{3}(P - (\mathbf{J} - \mathbf{I} - P))$ is PSD and thus yields an orthogonal embeding of P in S^4 so that $||i - j||^2 = 2 + \frac{2}{3}$ and a stem vector $\mathbf{b}$ satisfying $\mathbf{b}^2 = \frac{2 + \frac{2}{3}}{\frac{2}{3}} = 4$ Since $\alpha(P) = 4$ we have $\theta(P) = 4$. More examples will be explored in the exercises. for stem vectors for orthogonal representations of regular graphs.

6.1.1 Theorem. *(Lovász 1979) If $G = G(n,r)$ is an $r-$regular graph $r = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ then it admits an orthogonal representation for which there is a stem vector* $\mathbf{b}$ *such that:*

$$\mathbf{b}^2 \leq \frac{n\lambda_n}{\lambda_n - r}.$$

Proof Let $A_\beta = (A_G - \beta(\mathbf{J} - \mathbf{I}))$. Note that for any choice of β the smallest eigenvalue λ_{min} of A_β will be < 0 since $tr(A_\beta) = 0$. Let the eigenvectors of A_G be $\mathbf{j} = \mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ where $\mathbf{v_i}$ is the eigenvalue corresponding to λ_i. Let $\beta > 0$ and let $\lambda_{min} = -a$ be the smallest eigenvalue of A_β. The matrix $I + \frac{1}{a}A_\beta$ will be positive semi definite and thus $C = I + \frac{1}{a}A_\beta = M \cdot M^{tr}$. So for $(i,j) \notin E(G)$ we have $C_{i,j} = -\frac{\beta}{a} < 0$. But $C_{i,j} = <\mathbf{m_i}, \mathbf{m_j}>$, $<\mathbf{m_i}, \mathbf{m_i}> = 1$ hence we get an embedding $V(G) \longrightarrow S^d$ so that $(i,j) \notin E(G) \Longrightarrow ||\mathbf{m_i} - \mathbf{m_j}||^2 = 2 + \frac{2\beta}{a}$ hence by theorem 6.0.6 G will admit an orthogonal representation with a stem vector $\mathbf{b}$ satisfying $\mathbf{b}^2 = \frac{2+\frac{2\beta}{a}}{\frac{2\beta}{a}} = \frac{a+\beta}{\beta}$.

Since G is regular, $\mathbf{j}$ is an eigenvector and all other eigenvectors are orthogonal to it. This means that $\mathbf{J}\mathbf{v_i} = \mathbf{0}$; $2 \leq i \leq n$. We thus have:

$$A_\beta \mathbf{j} = (r - \beta(n-1))\mathbf{j} \qquad A_\beta \mathbf{v_i} = (\lambda_i + \beta)\mathbf{v_i}.$$

Hence $\lambda_{min} = min\{\beta + \lambda_n, r - \beta(n-1)\}$. If we select $\beta = \frac{r-\lambda_n}{n}$ then both will be equal and $a = -\lambda_{min} = -\lambda_n - \beta$.
And we get: $||\mathbf{b}||^2 = \frac{a+\beta}{\beta} = -\frac{-\lambda_n}{\frac{r-\lambda_n}{n}} = -\frac{n\lambda_n}{r-\lambda_n}$ ◇

Note first that this quantity can be computed in polynomial time. There are two caveats in this theorem. The first is an attempt to apply it to non regular graphs, and the second applying it to regular graphs and assuming that it is $= \vartheta(G)$. The exercises show such examples. As one may expect, this theorem does not give the best possible (shortest) stem vector. but for some interesting families of graphs it does. We shall explore these cases in the next chapter.

We conclude this section by introducing one of the classical definitions of the now famous Lovász function (actually, chronolgically it was the first one):

For a given orthogonal representation (G, ϕ) let $\mathbf{b}^2_\phi = min\{\mathbf{b}^2\}$ where the minimum is taken over all stem vectors of the orthogonal representation.

$$\vartheta(G) = min\{\mathbf{b}^2_\phi\}$$

where the minimum is over all orthogonal representations (G, ϕ).

$\vartheta(G)$ is the Lovász function. Clearly, $\vartheta(G) \geq \theta(G)$. Which leads us to numerous more questions, for example:

- Is $\vartheta(G) = \theta(G)$?
- Is it a better upper bound than $\alpha^*(G)$?
- Is it multiplicative?

In the next section we shall introduce equivalent, alternative definitons that will yeild more approaches to calculate this fascinating function and help us establish the Shannon capacity of more graphs. All definitions imply an optimization process. Our goals will be to find alternative definitions where some call for a maximizing process while others call for a minimizing process. This may "eliminate" the pain that can be sometimes associated with optimizing expressions. We encountered such an example when we used LP duality to prove that $\alpha^*(G \times H) = \alpha^*(G)\alpha^*(H)$.

Exercises

1. Show that $\overline{P}$ has an orthogonal representation with a stem vector b such that $||b||^2 = \frac{5}{2}$. (Hint: let the vectors have two coordinates: α, β. Choose them so they form an orthogonal representation and select an appropriate stem vector).
2. * The Kneser graphs $K_n(3m-1, m)$ are triangle free. Prove that $\theta(\overline{K_n(3m-1, m)}) < 3.1$ (in other words, if you wish to improve the upper bound on the Ramsey number $R(3; k)$ via the strong product these graphs will not help you). (hint: construct an orthogonal representation with the stem vector $(a, a, \ldots, a)$ for a properly selected a).
3. Prove that $\theta(L(K_{2n})) = n$ ($L(G)$ is the line graph of G).
4. Let $C_5 \oplus I_2$ be the prism over C_5. Prove that $\theta(C_5 \oplus I_2) = \sqrt{20}$ (so this will be another example of a connected graph for which the optimal word length is 2).
5. Show that $\theta(K_{1,n^2}) > \frac{n\lambda_n}{\lambda_n - \lambda_1}$.