The Shannon Capacity of Graphs and related topics

Moshe Rosenfeld

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Cvičení

In preparation for the next section we shall work out some of our old problems and introduce a couple of new ones.

Exercises

1. (a) Prove that $\alpha(C_7^2) = 10$

   Proof $\alpha^*(G_7) = \frac{7}{2}$ $\alpha(C_7) = 3$ hence $\alpha(C_7^2) \leq \lfloor\frac{21}{2}\rfloor = 10$. On the other hand, $\{(0,0),(0,2),(1,4),(2,6),(2,1),(3,3),(4,5),(4,1),(5,3),(6,5)\}$ is an independent set with 10 vertices. $\diamondsuit$

   (b) Prove that $\alpha(C_{4k+1}^2) = k(4k+1)$

   Proof see notes.

   (c) Prove that $\alpha(C_{4k+3}^2) = 4k^2 + 5k + 1$

   Proof Try to generalize the construction for C_7.

   * Let $\mathbf{A(n)} = \{(\mathbf{a_1}, \ldots, \mathbf{a_{2n}}) | \mathbf{a_i} \in \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4},\ \mathbf{a_{2i-1}} < \mathbf{a_{2i}}\ \mathbf{1} \geq \mathbf{i} \geq \mathbf{n}\}$. Let $\mathbf{B} \subset \mathbf{A(n)}$, $|\mathbf{B}| > \mathbf{4^n}$. Prove that there are two vectors b and c in $\mathbf{B}$ and a corresponding index k such that $\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\} = \emptyset$.

   Proof Fix a pair of coordinates $\{2i-1, 2i\}$ and consider all pairs of entries of all vectors of $\mathbf{B}$ that appears in these two coordinates. There must be an index $2i-1$ for which this set contains at least 5 distinct pairs for otherwise, $\mathbf{B}$ would contain at most 4^n vectors. But among 5 distinct pairs of numbers from $\{0,1,2,3,4\}$ there must be one disjoint pair. $\diamondsuit$

   * Prove that $\alpha(C_9^3) = 81$

   Proof See notes.

a. Let G be a triangle-free, 2-connected cubic graph. Prove that $\frac{|G|}{3} \leq \theta(G) \leq \frac{|G|}{2}$.

b. Prove that if $\theta(G) = \frac{|G|}{2}$ then G is bipartite.

** Let $\mathbf{A(n) = \{(a_1, \ldots, a_{2n}) | a_i \in 0, 1, 2, 3, 4,\ \ a_{2i-1} < a_{2i}\ 1 \geq i \geq n\}}$. Let $\mathbf{B \subset A(n)}$, $\mathbf{|B| > (\frac{5}{2})^n}$. Prove that there are two vectors b and c in $\mathbf{B}$ such that $|\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\}| \leq 1\ \forall k$.

*** Is $(\frac{5}{2})^n$ best possible?

Hints to selected exercises